Micromechanics and Homogenization Principles

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Nomenclature

$\varepsilon_{ij}^\ast$  eigenstrain
$\sigma_{ij}^\ast$  eigenstress
$\varepsilon_{ij}^c$  constraint (or total) strain
$\sigma_{ij}^c$  constraint stress
$\varepsilon_{ij}^I$  strain in inclusion
$\sigma_{ij}^I$  stress in inclusion (or elastic stress)
$\varepsilon_{ij}^{(inh)}$  strain in inhomogeneity
$\sigma_{ij}^{(inh)}$  stress in inhomogeneity
$\varepsilon_{ij}^M$  strain in matrix
$\sigma_{ij}^M$  stress in matrix
$\varepsilon_{ij}^A$  uniform applied strain, cf. section 3.3
$\sigma_{ij}^A$  stress inside the matrix under uniform applied strain $\varepsilon_{ij}^A$, cf. section 3.3
$\sigma_{ij}^{A\,(inh)}$  stress inside the inhomogeneity under uniform applied strain $\varepsilon_{ij}^A$, cf. section 3.3
$u_i^c$  Constrained displacement
$\langle \varepsilon_{ij} \rangle$  Average strain
$\langle \sigma_{ij} \rangle$  Average stress
$S_{ijkl}$  Eshelby’s tensor
$C_{ijkl}^{inh}$  Elastic constants of the inhomogeneity
$\lambda_M$  Lame’s constant of the matrix
$\mu_M$  Shear modulus of the matrix
$k_M$  Bulk modulus of the matrix
$\lambda_I$  Lame’s constant of the inclusion
$\mu_I$  Shear modulus of the inclusion
$k_I$  Bulk modulus of the inclusion
$\mu$  Effective shear modulus
$k$  Effective bulk modulus
\( \langle \varepsilon \rangle \) Average dilatational strain
\( \langle \varepsilon_I \rangle \) Uniform dilatational strain in Inclusion
\( U_0 \) Strain energy of equivalent homogeneous media
\( K_{23} \) Plain strain bulk modulus
\( A_{ijkl}^{\text{dil}} \) Dilute concentration tensor
\( T_{ijkl} \) Interaction tensor
\( E_M \) Elastic modulus of matrix
\( E_I \) Elastic modulus of Inclusion
\( E_f \) Elastic modulus of fiber
\( E_{ii} \) Uniaxial modulus
\( E \) Effective young’s modulus
\( \nu_I \) Poisson’s ratio of inclusion
\( \nu_f \) Poisson’s ratio of fiber
\( \nu_M \) Poisson’s ratio of matrix
\( c \) Volume fraction
\( \mu_{23} \) Transverse shear modulus
1. Basic elements and theorems of mechanics

1.1. Minimum theorems

There are two fundamental energy principles in linear elasticity theory which are very useful in mechanics of heterogeneous media: theorem of minimum potential energy and theorem of minimum complementary energy.

We consider a problem of static elasticity with body forces \(F_i(x_k)\) and following boundary conditions:

\[
\sigma_{ij} n_j = f_i \quad \text{on} \quad S_\sigma, \\
u_i = U_i \quad \text{on} \quad S_u.
\] (1.1)

Here \(S_\sigma\) and \(S_u\) are complementary parts of the surface of the body of volume \(V\) and \(n_j\) are the components of the unit outward normal to the surface.

A set of admissible displacement fields \(\hat{u}_i(x_k)\) consists of any continuous displacement field that satisfies the displacement boundary condition [1.2] but otherwise arbitrary chosen.

**Theorem of minimum potential energy:**

The functional of the potential energy

\[
U_\epsilon = \int_V (W(\epsilon_{ij}) - F_i u_i) \, dv - \int_{S_\sigma} f_i u_i \, ds
\] (1.3)

has an absolute minimum on the admissible displacement field which satisfies the equations of equilibrium.

Here \(W(\epsilon_{ij})\) is the strain energy.

Theorem of minimum potential energy can be written as

\[
\hat{U}_\epsilon - U_\epsilon \geq 0,
\] (1.4)

where \(\hat{U}_\epsilon\) is the functional [1.3] for any admissible displacement field \(\hat{u}_i(x_k)\).

The proof of the theorem is based on the positive definite character of the strain energy.

A set of admissible stress fields \(\hat{\sigma}_{ij}(x_k)\) consists of any continuous stress field (and derivative of the stress field is as well continuous) that satisfies the stress boundary condition [1.1] and equilibrium equations but otherwise arbitrary chosen.

**Theorem of minimum complementary energy:**

The functional of the complementary energy

\[
U_\sigma = \int_V W(\sigma_{ij}) \, dv - \int_{S_u} \sigma_i U_i \, ds
\] (1.5)

has an absolute minimum on the admissible stress field which satisfies the compatibility equations.

Here \(W(\sigma_{ij})\) is the strain energy expressed in terms of stresses.
Theorem of minimum complementary energy can be written as

\[ \hat{U}_\sigma - U_\sigma \geq 0, \]

(1.6)

where \( \hat{U}_\sigma \) is the functional 1.5 for any admissible stress field \( \hat{\sigma}_{ij}(x_k) \).

The proof of the theorem is based on the positive definite character of the strain energy (expressed in terms of stresses).
2. Eshelby’s inclusion

Consider a homogeneous solid of volume $V$ with elastic constants $C_{ijkl}$. Let us assume that the subvolume $V_0$ undergoes a volume/shape change due to the phase transformation, thermal expansion, etc. At this point of consideration, it is assumed that both the matrix (volume $V - V_0$) and the inclusion (subvolume $V_0$) have the same elastic stiffness $C_{ijkl}$. If the volume $V_0$ is not embedded in the surrounding matrix, it will assume a strain $\epsilon^*_ij$ and it will experience a zero stress. $\epsilon^*_ij$ is called eigenstrain which means self-strain under zero stress. Eshelby’s inclusion problem is aimed to find the solution for the stress, strain and displacement fields both in the inclusion and in the matrix.

![Figure 2.1.: Matrix with an inclusion.](image)

2.1. Eshelby’s formula (interaction energy of matrix-inclusion system)

To calculate the total elastic energy of the solid with an inclusion ("inclusion energy" for short), consisting of the energy stored inside the inclusion as well as the energy stored in the matrix, we can either use the work method (cf. section 2.5) or make an explicit volume integration of the energy stored in the body. In the latter case the Eshelby’s formula [1] derived in the current section allows to reduce the volume integration to a much simpler surface integration over the inclusion/matrix interface.

Consider a matrix without an inclusion and a matrix with an inclusion subjected to the surface tractions (similar considerations are valid for the case of the applied displacement field). The total elastic energy stored in the "matrix only" (Fig. 2.2b) is

$$U_0 = \frac{1}{2} \int_V \sigma^{0}_{ij} \epsilon^{0}_{ij} dV,$$

(2.1)

where $U_0$ is the total elastic energy stored in matrix, $\sigma^{0}_{ij}$ and $\epsilon^{0}_{ij}$ are the stress and strain induced in the matrix.

The total elastic energy stored in the matrix containing an inclusion (Fig. 2.2a) is given by

$$U = \frac{1}{2} \int_V \sigma_{ij} \epsilon_{ij} dV.$$

(2.2)
Figure 2.2.: Inclusion problem (a) and all matrix problem (b).

Considering Eqs. 2.1 and 2.2, the total elastic energy can be written as

\[ U = U_0 + \frac{1}{2} \int_V (\sigma_{ij} \epsilon_{ij} - \sigma^{0}_{ij} \epsilon^{0}_{ij})dV. \] (2.3)

For small strain setting, \( \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \) and

\[ \int_V \sigma_{ij} \epsilon_{ij}dV = \int_V \sigma_{ij} \left( \frac{1}{2}(u_{i,j} + u_{j,i}) \right) dV = \frac{1}{2} \int_V \left( (\sigma_{ij} u_i)_{,j} - \sigma_{ij,j} u_i + (\sigma_{ij} u_j)_{,i} - \sigma_{ij,i} u_j \right) dV \] (2.4)

Since in the absence of the body forces \( \sigma_{ij,i} = 0 \) and the stress tensor \( \sigma_{ij} \) is symmetric, the above equation can be rewritten as

\[ \int_V \sigma_{ij} \epsilon_{ij}dV = \int_V (\sigma_{ij} u_i)_{,j} dV = \] (2.5)

(applying Gauss theorem and taking into account that \( \sigma_{ij,n_j} = t_i \))

\[ = \int_S \sigma_{ij} u_i n_j dS = \int_S t_i u_i dS \] (2.6)

So Eq. 2.3 takes the form

\[ U = U_0 + \frac{1}{2} \int_S (t_i u_i - t^{0}_i u^{0}_i) dS. \] (2.7)

Consider now an auxiliary problem shown in Fig. 2.3. Body forces are used to mimic the inclusion-matrix interaction. Suppose that in the region outside the action of body forces, i.e. in the matrix, the elastic fields \( \tilde{\sigma}_{ij}, \tilde{\epsilon}_{ij}, \tilde{u}_{ij} \) corresponding to the problem in Fig. 2.3 are identical to the elastic fields \( \sigma_{ij}, \epsilon_{ij}, u_{ij} \) in the initially sought problem in Fig. 2.2a.
Principle of superposition is used to calculate the interaction energy between the matrix and inclusion.

The equivalent stress, strain and displacement fields are given by

$\hat{\sigma}_{ij} = \sigma_{ij}^0 + \sigma_{ij}', \quad (2.8)$

$\hat{\epsilon}_{ij} = \epsilon_{ij}^0 + \epsilon_{ij}', \quad (2.9)$

$\hat{u}_i = u_i^0 + u_i', \quad (2.10)$

where $\sigma_{ij}^0, \epsilon_{ij}^0$ and $u_i^0$ are the stresses, strains and displacements induced as a result of the tractions applied on the surface of the matrix without the inclusion (all matrix) - this problem identical to those shown in Fig. 2.2b.

$\sigma_{ij}', \epsilon_{ij}'$ and $u_i'$ are the stresses, strains and displacements induced in the matrix where the action of the inclusion is replaced by the body forces.

The equivalent elastic energy $\hat{U}$ is given as

$\hat{U} = \frac{1}{2} \int_V (\sigma_{ij}^0 + \sigma_{ij}') (\epsilon_{ij}^0 + \epsilon_{ij}') dV, \quad (2.11)$

$\hat{U} = U_0 + U' + \frac{1}{2} \int_V (\sigma_{ij}^0 \epsilon_{ij} + \sigma_{ij}' \epsilon_{ij}) dV,$

$\sigma_{ij}' \epsilon_{ij} = C_{ijkl} \epsilon_{kl} \epsilon_{ij} = \sigma_{ij}' \epsilon_{ij}, \quad (2.13)$

$U_{\text{INT}} = \int_{V_1} \sigma_{ij}^0 \epsilon_{ij} dV + \int_{V_{II}} \sigma_{ij}^0 \epsilon_{ij} dV = \int_{\Sigma} t_i^0 u_i dS + \int_S t_i' u_i dS - \int_{\Sigma} t_i' u_i^0 dS, \quad (2.14)$

As the surface of inclusion is not subjected to traction force, $t_i' u_i^0 = 0$. The interaction energy will be equal to

$U_{\text{INT}} = \int_{\Sigma} (t_i^0 u_i' - t_i' u_i^0) dS. \quad (2.15)$
In region $V_1$: $\sigma_{ij}^0 = 0$, $\sigma_{ij}' = 0$.
In region $V_{II}$: $\sigma_{ij}' = 0$, $\sigma_{ij}^0 = 0$, but $\sigma_i^0 = 0$, $\sigma_i' = 0$.
As the surface of inclusion is not subjected to traction force, $t_i'u_i^0 = 0$. The interaction energy will be equal to
\[
U_{\text{INT}} = \int_S (t_i' u_i - t_i u_i^0) dS.
\] (2.16)
and, finally,
\[
U = U_0 + \frac{1}{2} U_{\text{INT}} = U_0 + \frac{1}{2} \int_S (t_i' u_i - t_i u_i^0) dS.
\] (2.17)

**Example**

The beam of thickness $t$ and width $w$ as shown in the figure below is made of the material with the Young’s modulus $E_M$ which has a built-in inclusion with the Young’s modulus $E_I$. The beam is fixed at one end and the tensile stress $\sigma$ is applied at the other end. Calculate the total elastic energy.
Solution:
The total energy stored in the beam is given by Eq. 2.17 as

\[ U = U_0 + \frac{1}{2} \int_S (t^0_i u_i - t_i u^0_i) dS. \]  

(2.18)

Energy \( U_0 \) stored inside the homogeneous matrix ("M") material (supposing that all the beam is now made only of the matrix material) is equal to

\[ U_0 = \frac{1}{2} \int_V \sigma_{ij}^0 \varepsilon_{ij}^0 dV = \frac{1}{2} \sigma \epsilon V = \frac{1}{2} \sigma \epsilon (a + b) wt = \left\{ \epsilon = \frac{\sigma}{E_M} \right\} = \frac{\sigma^2}{2E_M} (a + b) wt, \]

(2.19)

where \( wt \) is the cross-sectional area. Please recall that the stress in each cross-section (either the matrix or the inclusion) is equal to the externally applied one.
Displacement of the surface of the inclusion in the "all-matrix" case is

\[ u^0 = \frac{\sigma}{E_M} a, \]

while in the case of the matrix with the inclusion, it is

\[ u = \frac{\sigma}{E_1} a. \]

Substituting all the expressions in Eq. 2.18 gives

\[ U = \frac{\sigma^2}{2E_M} (a + b) wt + \int_S \left( \sigma_{ij} - \frac{\sigma}{E_1} a - \frac{\sigma}{E_M} a \right) dS = \frac{\sigma^2}{2E_M} (a + b) wt + \left( \frac{\sigma^2}{E_1} - \frac{\sigma^2}{E_M} \right) awt \]

\[ = \frac{bwt - \sigma^2}{2E_M} + awt \frac{\sigma^2}{2E_1}. \]

2.2. Eshelby’s approach

Eshelby used principle of superposition and Green’s function to solve the internal stress field. Eshelby followed the following steps of virtual experiment to solve the stress, strain and displacement field both in the inclusion and in the matrix.

Step 1:
Remove the region of interest or the inclusion from the matrix and allow the unconstrained transformation to take place.

<table>
<thead>
<tr>
<th>matrix</th>
<th>inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_{ij} = 0 )</td>
<td>( e_{ij} = e_{ij}^* )</td>
</tr>
<tr>
<td>( \sigma_{ij} = 0 )</td>
<td>( \sigma_{ij} = 0 )</td>
</tr>
<tr>
<td>( u_i = 0 )</td>
<td>( u_i = e_{ij}^* x_j )</td>
</tr>
</tbody>
</table>
Step 2:
In this step, surface traction is applied to the inclusion to make the inclusion return to original shape.

\[
\begin{array}{c|c}
\text{matrix} & \text{inclusion} \\
\hline
\varepsilon_{ij} = 0 & \varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^* = 0 \\
\sigma_{ij} = 0 & \sigma_{ij} = -C_{ijkl} \varepsilon_{kl}^{el} \\
u_i = 0 & u_i = 0 \\
\end{array}
\]

Step 3:
Put the inclusion back in to the matrix. Now the traction applied which can now be interpreted as the internal body forces. Therefore there is no change in stress or strain in the matrix.
**Figure 2.8.** Step 3 of Eshelby’s approach.

**Step 4:**

Remove the traction force. The change from the step 3 is obtained by applying body force in direction opposite to the applied traction to the surface of the inclusion.

**Figure 2.9.** Step 4 of Eshelby’s approach.

<table>
<thead>
<tr>
<th>matrix</th>
<th>inclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{ij} = e^c_{ij}$</td>
<td>$e_{ij} = e^c_{ij}$</td>
</tr>
<tr>
<td>$\sigma_{ij} = \sigma^c_{ij}$</td>
<td>$\sigma_{ij} = \sigma^c_{ij} - \sigma^<em><em>{ij} = C</em>{ijkl}(e^c_{kl} - e^</em>_{kl})$</td>
</tr>
<tr>
<td>$u_i = u_i^c$</td>
<td>$u_i = u_i^c$</td>
</tr>
</tbody>
</table>

Let $u_i^c$ be the displacement field in response to the body force $b_j$ on the surface of inclusion. It can be expressed in terms of the Green’s function of the elastic body.

**Note:** The elastic Green’s function gives the displacement in the i-direction at x due to a point force acting in the j-direction at $x'$.

$$b_j = -T_j = \sigma^*_{jk} n_k$$  \hspace{1cm} (2.20)
\[ u^c_i = \int_{S_0} b_j(x')G_{ij}(x,x')dS(x') = \int_{S_0} \sigma^c_{jk}n_k(x')G_{ij}(x,x')dS(x') \] (2.21)

The displacement gradient is given as

\[ u^c_{ij} = \int_{S_0} \sigma^c_{lk}n_k(x')(G_{il,j}(x,x') + G_{jl,i}(x,x'))dS(x') \] (2.22)

\[ e^c_{ij} = \frac{1}{2}(u^c_{i,j} + u^c_{j,i}), \] (2.23)

\[ e^c_{ij} = \frac{1}{2} \int_{S_0} \sigma^c_{lk}n_k(x')(G_{il,j}(x,x')dS(x') + G_{jl,i}(x,x'))dS(x'). \] (2.24)

\[ \sigma^c_{ij}(x) = C_{ijkl}e^c_{kl} \] (2.25)

In order to obtain the explicit expression for the stresses and strains everywhere, the constrained strain field must be determined both inside and outside the inclusion. A fourth order tensor \( S_{ijkl} \) is used to relate the constrained strain inside the inclusion to its eigenstrain

\[ e^c_{ij} = S_{ijkl}e^c_{kl}, \] (2.26)

Where \( S_{ijkl} \) is referred to as Eshelby tensor.

For an ellipsoidal inclusion in homogeneous infinite matrix, the Eshelby tensor is constant tensor. Consequently the stress and strain field inside the inclusion are uniform.

### 2.3. Eshelby’s tensor in isotropic medium

For isotropic medium, the Eshelby’s tensor for an ellipsoidal inclusion with the semi-axes \( a > b > c \) (the semi axis \( a \) aligns with the coordinate \( x \), \( b \) with \( y \) and \( c \) with \( z \)) can be expressed in terms of elliptic integrals:

\[ S_{1111} = \frac{3}{8\pi(1-\nu)}a^2I_{11} + \frac{1-2\nu}{8\pi(1-\nu)}I_1, \]
\[ S_{1122} = \frac{1}{8\pi(1-\nu)}b^2I_{12} + \frac{1-2\nu}{8\pi(1-\nu)}I_1, \]
\[ S_{1133} = \frac{1}{8\pi(1-\nu)}c^2I_{13} + \frac{1-2\nu}{8\pi(1-\nu)}I_1, \]
\[ S_{1212} = \frac{1}{16\pi(1-\nu)} I_{12} + \frac{1-2\nu}{16\pi(1-\nu)}(I_1 + I_2), \]
\[ S_{1112} = 0, \quad S_{1223} = 0, \quad S_{1321} = 0. \] (2.27)

The \( I \) terms are defined via standard elliptic integrals as

\[ I_1 = \frac{4\pi abc}{(a^2 - b^2)(a^2 - c^2)^{1/2}} \left[ F(\theta, k) - E(\theta, k) \right], \] (2.28)
\[ I_3 = \frac{4\pi abc}{(b^2 - c^2)(a^2 - c^2)^{1/2}} \left[ \frac{b(a^2 - c^2)^{1/2}}{ac} - E(\theta, k) \right], \] (2.29)
where

\[ \theta = \arcsin \sqrt{\frac{a^2 - c^2}{a^2}}, \quad k = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}} \]  

(2.30)

and

\[ I_1 + I_2 + I_3 = 4\pi, \]

\[ 3I_{11} + I_{12} + I_{13} = \frac{4\pi}{a^2}, \]

\[ 3a^2 I_{11} + b^2 I_{12} + c^2 I_{13} = 3I_1, \]  

\[ I_{12} = \frac{I_2 - I_1}{a^2 - b^2} \]  

(2.31)

and the incomplete elliptic integral of the first kind \( F \) and the incomplete elliptic integral of the second kind \( E \) are

\[ F(\theta, k) = \int_0^\theta \frac{dw}{(1 - k^2 \sin^2 w)^{1/2}}, \quad E(\theta, k) = \int_0^\theta (1 - k^2 \sin^2 w)^{1/2} dw. \]  

(2.32)

**Eshelby’s tensor for spherical inclusion in isotropic medium**

For a spherical inclusion \((a = b = c)\), Eshelby’s tensor has the following elegant expression:

\[ S_{ijkl} = \frac{5\nu - 1}{15(1 - \nu)} \delta_{ij} \delta_{kl} + \frac{4 - 5\nu}{15(1 - \nu)} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \]  

(2.33)

The Eshelby’s tensor does not depend on the radius of the inclusion.

### 2.4. Simple 1D spring analogy to illustrate eigenstrain

To illustrate eigenstrain, constraint strains and strains in inclusion and matrix consider very simple 1D analogy with a spring:

The spring is initially placed between two metallic clamps so both the spring and the outer cage are stress/strain free.

Let’s suppose that the spring due to some reasons, e.g. thermal expansion, is now 1% longer. If we take the spring out of the cage (matrix), the eigenstrain \( \epsilon_{ij}^1 = 1\% \).

Now put the spring back to the cage. We will have some interaction between the spring and the cage. It is clear that the spring will be constrained by the cage and it will not attain 1% of
deformation. Instead of this, both the spring and the cage will experience deformations. Let’s suppose that the constrained strain is $\epsilon_{ij}^c = 0.3\%$ (we measure it from the initial position). This means that the interface between the spring and the matrix has moved by $0.3\%$: $\epsilon_{ij}^M = \epsilon_{ij}^c = 0.3\%$. And the real strain within the inclusion is $\epsilon_{ij}^I = \epsilon_{ij}^c - \epsilon_{ij}^* = 0.3\% - 1\% = -0.7\%$.

This shows that the inclusion is compressed (the constrained strain has not yet reached or exceeded the eigenstrain). The constrained strain is positive since we measure here the strain in reference to the initial state (when no transformation to the spring has yet occurred).

In our example and with no external loading applied $\epsilon_{ij}^* = \max\{\epsilon_{ij}^c, |\epsilon_{ij}^I|, \epsilon_{ij}^*\}$. It should be noticed that the eigenstrain is constant (it corresponds to the occurred transformation of the inclusion) while the constrained strain as well as strains in the inclusion and matrix will be changing with applied external loading.

In case of $\epsilon_{ij}^c \approx 0$ (very stiff matrix or soft inclusion) $\epsilon_{ij}^I \approx -\epsilon_{ij}^*$. The inclusion will experience zero stresses when $\epsilon_{ij}^I = \epsilon_{ij}^c - \epsilon_{ij}^* = 0$. From this follows $\epsilon_{ij}^I = \epsilon_{ij}^*$, i.e. the inclusion (and matrix) is stretched to the transformation shape when no surrounding matrix is considered (step I of virtual experiment done by Eshelby).

### 2.5. Inclusion/matrix energy calculation by the work done to the system

There are two ways to calculate the energy stored in the inclusion-matrix system:

- integrate the strain energy density over the inclusion + matrix volume
- calculate the work done to the inclusion-matrix system. It will be equal to the accumulated potential energy of the composite.

Consider now the second approach at each of four steps of Eshelby’s virtual experiment.

The work done during the Step 1 is zero since no force is applied to the structure. The deformation is due to eigenstrains.

During the Step 2 the compressive forces $T_i = -\sigma_{ji}^* n_j$ are applied to the inclusion surface. The corresponding displacements are as well negative and are calculated as $u_i = -\epsilon_{kj}^* x_k$. Both tractions and displacements are functions of coordinates: $T_i = T_i(\bar{x})$, $u_i = u_i(\bar{x})$. It should be noticed that although the eigenstrains $\epsilon_{kj}^*$ are constant in the ellipsoidal inclusion, the displacements of the inclusion-matrix interface are a function of the position at the interface.
The work done by the end of Step 2) is equal to
\[
W_{1\rightarrow2} = \frac{1}{2} \int_{S_0} T_i u_i \, ds = \frac{1}{2} \int_{S_0} (-\sigma_{ji}^* n_j)(-\epsilon_{ki}^* x_k) \, ds = \frac{1}{2} \sigma_{ji}^* \epsilon_{ki}^* \int_{S_0} x_k n_j \, ds = \frac{1}{2} \sigma_{ji}^* \epsilon_{ki}^* \int_{V_0} x_{k,j} \, dv =
\]
\[
= \frac{1}{2} \sigma_{ji}^* \epsilon_{ki}^* \int_{V_0} \delta_{kj} \, dv = \frac{1}{2} \sigma_{ij}^* \epsilon_{ij}^* V_0 \tag{2.34}
\]
The subscript in "W_{1\rightarrow2}" should be understood as "the work done from the end of the Step 1 till the end of the Step 2".
There is no work \(W_{2\rightarrow3}\) done during the Step 3.
When we consider Step 4 the only change in the applied loading to the system inclusion-matrix is the retrieval of the force applied during the Step 2. So, the applied force changes from \(T_i = -\sigma_{ji}^* n_j\) to zero. However, the displacement is not \(u_i = -\epsilon_{ki}^* x_k\) as in Step 2) but \(u_i^c\).
The work done to the inclusion + matrix by the end of Step 4) is equal to
\[
W_{3\rightarrow4} = \frac{1}{2} \int_{S_0} T_i u_i^c \, ds = \frac{1}{2} \int_{S_0} (-\sigma_{ji}^* n_j) u_i^c \, ds = -\frac{1}{2} \sigma_{ji}^* \int_{S_0} u_i^c n_j \, ds =
\]
\[
= -\frac{1}{2} \sigma_{ji}^* \int_{V_0} u_{i,j} \, dv = -\frac{1}{2} \sigma_{ij}^* \epsilon_{ij}^* V_0 \tag{2.35}
\]
Let’s sum up the work done to the entire system at all four steps:
\[
E = E_0 + W_{0\rightarrow1} + W_{1\rightarrow2} + W_{2\rightarrow3} + W_{3\rightarrow4} =
\]
\[
= \frac{1}{2} \sigma_{ij}^* \epsilon_{ij}^* V_0 - \frac{1}{2} \sigma_{ij}^1 \epsilon_{ij}^1 V_0 = \frac{1}{2} \sigma_{ij}^* \epsilon_{ij}^* V_0 - \frac{1}{2} \sigma_{ij}^c \epsilon_{ij}^c V_0 = -\frac{1}{2} (\sigma_{ij}^{c} - \sigma_{ij}^*) \epsilon_{ij}^* V_0 = -\frac{1}{2} (\sigma_{ij}^{c} - \sigma_{ij}^*) \epsilon_{ij}^* V_0 = \frac{1}{2} \sigma_{ij}^1 \epsilon_{ij}^1 V_0 \tag{2.36}
\]
Formula \(2.36\) is the total elastic energy of the system after all four steps.

Now we will apply the same approach to calculate the work done separately to the inclusion and the matrix. Afterwards we will calculate the elastic energy only in the inclusion or in the matrix which is accumulated after all four steps.
Since the inclusion and matrix are considered separately up to Step 4, the work done to the inclusion will coincide with the results obtained above, namely:
\[
W^I_{1\rightarrow2} = \frac{1}{2} \sigma_{ij}^* \epsilon_{ij}^* V_0, \tag{2.37}
\]
\(W^I_{0\rightarrow1} = 0\) and \(W^I_{2\rightarrow3} = 0\).
There is no work done to the matrix during Steps 1 to 3.
The difference between the matrix-inclusion system and separate consideration of the inclusion is that the matrix also exerts force on the inclusion and, consequently, there is a corresponding part of the work done to the inclusion. So some energy is transferred from the inclusion to the matrix. The forces acting on the inclusion and on the matrix at the end of Step 4:
\[
(\sigma_{ij}^c - \sigma_{ij}^*) n_i - \text{force on the inclusion},
\]
\[-(\sigma_{ij}^c - \sigma_{ij}^*) n_i - \text{force on the matrix}. \tag{2.38}\]
The work done to the inclusion by the end of Step 4) is equal to
\[
W^I_{3\rightarrow4} = \frac{1}{2} \int_{S_0} (\sigma^c_{ij} - 2\sigma^*_{ij}) n_i u_j^c \, ds = \frac{1}{2} (\sigma^c_{ij} - 2\sigma^*_{ij}) \epsilon^c_{ij} V_0. \tag{2.39}
\]

The work done to the matrix by the end of Step 4) is equal to
\[
W^M_{3\rightarrow4} = -\frac{1}{2} \int_{S_0} (\sigma^c_{ij} - \sigma^*_{ij}) n_i u_j^c \, ds = -\frac{1}{2} (\sigma^c_{ij} - \sigma^*_{ij}) \epsilon^c_{ij} V_0. \tag{2.40}
\]

Elastic energy in the inclusion at the end of Step 4:
\[
E^I = \frac{1}{2} \sigma^c_{ij} \epsilon^c_{ij} V_0 + \frac{1}{2} (\sigma^*_{ij} - 2\sigma^*_{ij}) \epsilon^c_{ij} V_0 = \frac{1}{2} (\sigma^c_{ij} \epsilon^c_{ij} + \sigma^c_{ij} \epsilon^c_{ij} - 2\sigma^*_{ij} \epsilon^c_{ij}) V_0
= \frac{1}{2} (\sigma^c_{ij} - \sigma^*_{ij}) (\epsilon^c_{ij} - \epsilon^c_{ij}) V_0 = \frac{1}{2} \sigma^I_{ij} \epsilon^I_{ij} V_0 \tag{2.41}
\]

Elastic energy in the matrix at the end of the Step 4:
\[
E^M = -\frac{1}{2} (\sigma^c_{ij} - 2\sigma^*_{ij}) \epsilon^c_{ij} V_0 = -\frac{1}{2} \sigma^M_{ij} \epsilon^M_{ij} V_0 \tag{2.42}
\]

Summing the Eq. 2.41 and 2.42 we get the total energy of the inclusion-matrix system:
\[
E = E^I + E^M = \frac{1}{2} \sigma^I_{ij} \epsilon^I_{ij} V_0 - \frac{1}{2} \sigma^I_{ij} \epsilon^I_{ij} V_0 = \frac{1}{2} \sigma^I_{ij} (\epsilon^I_{ij} - \epsilon^I_{ij}) V_0 = -\frac{1}{2} \sigma^I_{ij} \epsilon^I_{ij} V_0 \tag{2.43}
\]

Eq. 2.43 is exactly the same as Eq. 2.36.
Consider now one of the simplest but very important example of spherical inclusion.

Figure 2.10.: CAE model of the spherical inclusion.

Let the radius of the inclusion be R. Suppose that only a shear eigenstrain $\epsilon_{12}^* = \epsilon$ is nonzero.

Calculate the total elastic energy:

$$ E = -\frac{1}{2}\sigma_{ij}^c \epsilon_{ij} V_0 = -\frac{1}{2}(\sigma_{ij}^c - \sigma_{ij}^* \epsilon_{ij}^* V_0$$

consider that $\epsilon_{12}^* = \epsilon_{21}^* = \epsilon = -(\sigma_{12}^c - \sigma_{12}^*)\epsilon V_0$. (2.44)

For isotropic material

$$\sigma_{12}^c = 2\mu \epsilon_{12}^c \text{ and } \sigma_{12}^* = 2\mu \epsilon_{12}^* = 2\mu \epsilon. \quad (2.45)$$

The necessary components of Eshelby’s tensor:

$$S_{1212} = S_{1221} = \frac{4 - 5\nu}{15(1 - \nu)}.$$

With the help of Eshelby’s tensor (Formula 2.46) the constrained strain is defined as

$$\epsilon_{12}^c = \frac{4 - 5\nu}{15(1 - \nu)}(\epsilon_{12}^* + \epsilon_{21}^*) = \frac{4 - 5\nu}{15(1 - \nu)}2\epsilon. \quad (2.47)$$

Substitution of Eq. 2.47 in Eq. 2.45 leads to

$$\sigma_{12}^c = \frac{4 - 5\nu}{15(1 - \nu)}4\mu \epsilon. \quad (2.48)$$

Volume of a sphere inclusion of radius R: $V_0 = \frac{4}{3}\pi R^3$.

Thus, Eq. 2.44 obtains the form

$$E = -\left(\frac{4 - 5\nu}{15(1 - \nu)}4\mu \epsilon - 2\mu \epsilon \right)\frac{4}{3}\pi R^3 = \frac{8}{45} \frac{7 - 5\nu}{1 - \nu} \mu \epsilon^2 \pi R^3. \quad (2.49)$$
Define now the total elastic energy of the system if a uniform stress field $\sigma_{12}^A = \tau$ is applied to the solid.

Using Colonetti’s theorem\(^1\), we can write the total energy as

$$ E^T = E + E^A. \quad (2.50) $$

The energy $E^A$ resulting from the external stress can be calculated as

$$ E^A = \frac{1}{2} \left( \sigma_{12}^A \epsilon_{12}^A + \sigma_{21}^A \epsilon_{21}^A \right) V = \sigma_{12}^A \epsilon_{12}^A V. \quad (2.51) $$

$$ \epsilon_{12}^A = \frac{\sigma_{12}^A}{2\mu} = \frac{\tau}{2\mu}. \quad (2.52) $$

\(^1\)Colonetti’s theorem states that there is no interference term in the expression of the total elastic energy between the dislocation strain field and the homogeneous strain (which can be considered as resulting from an external stress).
Substituting Eq. 2.52 in Eq. 2.51, we come to the final expression for the energy $E^\Lambda$:

$$E^\Lambda = \frac{\tau^2}{2\mu} V.$$  (2.53)

And the total energy (Formula 2.50) takes the final form

$$E^T = E + E^\Lambda = \frac{8}{45} \frac{7 - 5\nu}{1 - \nu} \mu \epsilon^2 \pi R^3 + \frac{\tau^2}{2\mu} V.$$  (2.54)

Figure 2.12.: $\sigma_{12}$ stress in the cross-section of the inclusion as a result of eigenstrains (corresponding eigenstress $\sigma^*_i = 100$ MPa) and external shear stresses $\sigma_{12} = 100$ MPa (deformation magnified 100 times).

Define now the enthalpy of the system $H$ and the driving force for inclusion growth $f(R) = \frac{\partial H(R)}{\partial R}$.

$$H = E - E^\Lambda - \sigma^*_i \epsilon^*_i \epsilon_0 = \frac{8}{45} \frac{7 - 5\nu}{1 - \nu} \mu \epsilon^2 \pi R^3 - \frac{\tau^2}{2\mu} V - \frac{8}{3} \tau \epsilon \pi R^3.$$  (2.55)
The driving force for the inclusion growth:

\[
f(R) = -\frac{\partial H(R)}{\partial R} = -\frac{8}{15} \frac{7 - 5\nu}{1 - \nu} \mu \varepsilon^2 \pi R^2 + 8\tau\varepsilon\pi R^2 = 8\pi R^2 \varepsilon \left( \tau - \frac{7 - 5\nu}{15(1 - \nu)} \mu \varepsilon \right).
\]  

(2.56)
3. Eshelby’s inhomogeneity

3.1. Example of a void filled with the liquid to a certain pressure

Eshelby’s solution can be applied to a problem of an inhomogeneity, crack, etc. A technique in which the eigenstrain in the inclusion is chosen to mimic the behavior of an inhomogeneity, crack, ... is called the equivalent inclusion method. The method is applied to the ellipsoidal inhomogeneities since the stress-strain field is constant inside an ellipsoidal inclusion.

Let’s illustrate an Eshelby’s equivalent inclusion method on a very simple example of a cavity $V_0$ filled with a liquid with a pressure $p_0$.

We replace the liquid with an inclusion. The eigenstrain in the inclusion must excite the stress field in the inclusion which is exactly the same as in the liquid\(^1\):

$$\sigma_{ij}^I = -p_0 \delta_{ij}. \tag{3.1}$$

Our aim is to define the eigenstrain of the equivalent inclusion $\epsilon_{ij}^*$. Using Eshelby’s tensor $S_{ijkl}$ we can rewrite the expression for the inclusion stress in the form

$$\sigma_{ij}^I = \sigma_{ij}^c - \sigma_{ij}^* = C_{ijkl}(\epsilon_{kl}^c - \epsilon_{kl}^*) = C_{ijkl}(S_{klmn}\epsilon_{mn}^* - \epsilon_{kl}^*). \tag{3.2}$$

Consequently,

$$C_{ijkl}(S_{klmn} - \delta_{km}\delta_{ln})\epsilon_{mn}^* = -p_0 \delta_{ij}. \tag{3.3}$$

Eq. 3.3 is a system of six linear algebraic equations with respect to six unknown equivalent eigenstrains $\epsilon_{ij}^*$. The displacements of the surface $S_0$ are determined as

$$u_i = u_c^i = S_{ijkl}\epsilon_{kl}^* x_j. \tag{3.4}$$

Elastic energy in the matrix is defined by Eq. 2.42. It is the same when there is the liquid in the void and for the equivalent inclusion:

$$E^M = -\frac{1}{2} \sigma_{ij}^I \epsilon_{ij}^* V_0 = -\frac{1}{2} (-p_0 \delta_{ij}) S_{ijkl}\epsilon_{kl}^* V_0 = \frac{1}{2} p_0 S_{ijkl}\epsilon_{kl}^* V_0. \tag{3.5}$$

\(^1\)It is known that stress in a liquid is constant
3.2. Inhomogeneity with eigenstrain transformation

We can apply an Eshelby’s equivalent inclusion method to the transformed inhomogeneity problem. The difference between the transformed inhomogeneity and the transformed inclusion is that the inclusion has the same material properties as the matrix, whereas the material properties of the inhomogeneity are different from the matrix. We mark the elastic constants and other properties of the inhomogeneity with superscript " (inh)", e.g. \( C_{ijkl}^{(inh)} \).

The inhomogeneity experiences a permanent transformation described by its eigenstrain \( \epsilon_{ij}^{(inh)} \). We want to determine the stress-strain distribution in the solid as well as its total elastic energy. Compared with the problem of a void filled with the liquid to a certain pressure (section 3.1), the current problem is more complicated since inhomogeneity is a solid.

When we replace the inhomogeneity with an equivalent inclusion, the displacement and the traction force should match on the interface \( S_0 \). To satisfy these condition, it is sufficient that the elastic stress and the total strain are the same inside the transformed inhomogeneity and the equivalent inclusion:

\[
\epsilon_{ij}^{c(\text{inh})} = \epsilon_{ij}^{c} (= S_{ijkl} \epsilon_{kl}^{s}), \quad (3.6)\\
\sigma_{ij}^{(inh)} = \sigma_{ij}, \quad (3.7)
\]
In Eq. 3.7
\[ \sigma_{ij}^{(inh)} = \sigma_{kl}^{(inh)} - \sigma_{kl}^{\star} = C_{ijkl}^*(\epsilon_{kl}^{(inh)} - \epsilon_{kl}^{\star}), \]  
(3.8)
\[ \sigma_{ij}^l = \sigma_{kl}^l - \sigma_{kl}^\star = C_{ijkl}(\epsilon_{kl}^c - \epsilon_{kl}^\star) = (C_{ijkl}S_{klmn} - C_{ijmn})\epsilon_{mn}^\star. \]  
(3.9)
Substituting Eqs. 3.8 and 3.9 in Eq. 3.7 and taking into account the equality 3.6, we obtain
\[ C_{ijkl}^{(inh)}(\epsilon_{kl}^c - \epsilon_{kl}^{\star}) = C_{ijkl}(\epsilon_{kl}^c - \epsilon_{kl}^{\star}). \]  
(3.10)
Rewriting the last equation in the form
\[ \left((C_{ijkl}^{(inh)} - C_{ijkl})S_{klmn} + C_{ijmn}\right)\epsilon_{mn}^\star = C_{ijkl}^{(inh)}\epsilon_{kl}^{\star}, \]  
(3.11)
allows us to determine the eigenstrain \( \epsilon_{mn}^\star \) in the equivalent inclusion in terms of \( \epsilon_{kl}^{\star} \).
The elastic energy in the matrix for the transformed inhomogeneity problem is equal to the one for the equivalent inclusion problem:
\[ E_M = -\frac{1}{2}\sigma_{ij}^l\epsilon_{ij}^cV_0. \]  
(3.12)
The important fact is that the elastic energy in the transformed inhomogeneity is not equal to the elastic energy in the equivalent inclusion, namely
\[ E^{(inh)} = \frac{1}{2}\sigma_{ij}^{(inh)}\epsilon_{ij}^{(inh)}V_0 = \frac{1}{2}\sigma_{ij}^l(\epsilon_{ij}^c - \epsilon_{ij}^{\star})V_0 = \frac{1}{2}\sigma_{ij}^l(\epsilon_{ij}^c - \epsilon_{ij}^{\star})V_0, \]  
(3.13)
\[ E^I = \frac{1}{2}\sigma_{ij}^I\epsilon_{ij}^cV_0 = \frac{1}{2}\sigma_{ij}^l(\epsilon_{ij}^c - \epsilon_{ij}^{\star})V_0, \]  
(3.14)
and the same applies to the total energy:
\[ \text{inhomogeneity: } E = E^{(inh)} + E^M = -\frac{1}{2}\sigma_{ij}^l\epsilon_{ij}^{\star}V_0, \]  
(3.15)
\[ \text{inclusion: } E = E^I + E^M = -\frac{1}{2}\sigma_{ij}^l\epsilon_{ij}^{\star}V_0. \]  
(3.16)

### 3.3. Inhomogeneity without eigenstrain under applied uniform loading

Consider an inhomogeneity which does not have an eigenstrain. The inhomogeneity-matrix system is subjected to the action of external loading. We study the case when the loading is uniform. In case of uniform loading the stress-strain field should be uniform inside the inclusion-matrix system (the properties of the inhomogeneity are the same as of the matrix). But it is not the case when we consider inhomogeneity. So, we have to define the disturbed stress-strain field for a solid with inhomogeneity.

To find a solution of the problem, we will split it into two subproblems:

- The solid with inhomogeneity is under uniform strain \( \epsilon_{ij}^A \) \( \epsilon_{ij}^A \) corresponds to a strain in the homogeneous solid which has elastic constants \( C_{ijkl} \) of the matrix). In this case the stress field inside the matrix is \( \sigma_{ij}^A = C_{ijkl}\epsilon_{ij}^A \) and the stress field inside the inhomogeneity is \( \sigma_{ij}^{(inh)} = C_{ijkl}\epsilon_{ij}^{(inh)} \). They do not match. To assure the equilibrium, the body force \( \sigma_{ij}^{(inh)} - \sigma_{ij}^A \) must be applied to the surface \( S_0 \) of the inhomogeneity.
In the supplementary problem we have to remove this body force, i.e. to apply \(-\sigma^\text{inh}_{ij} \) to the surface \( S_0 \) of the inhomogeneity without taking into account the external loading. Denote the stresses under such loading as \( \sigma^c_{ij} \) and the strains as \( \epsilon^c_{ij} \).

Now superimpose these two subproblems. The elastic stresses inside the inhomogeneity are

\[
\sigma^{\text{inh}}_{ij} = \sigma^A_{ij} + \sigma^c_{ij} = C^{\text{inh}}_{ijkl} (\epsilon^A_{ij} + \epsilon^c_{ij}). \tag{3.17}
\]

Since in this problem the eigenstrain of the inhomogeneity is equal to zero \( \epsilon^*_{ij} = 0 \) the total strain is equal to the elastic strain inside the inhomogeneity:

\[
\epsilon^{\text{inh}}_{ij} = \epsilon^A_{ij} + \epsilon^c_{ij}. \tag{3.18}
\]

The elastic stress field in the equivalent inclusion (the equivalent inclusion has an eigenstrain \( \epsilon^*_{ij} \)) is calculated as

\[
\sigma^I_{ij} = \sigma^A_{ij} + \sigma^c_{ij} - \sigma^*_{ij} = C^{\text{inh}}_{ijkl} (\epsilon^A_{ij} + \epsilon^c_{ij} - \epsilon^*_{ij}). \tag{3.19}
\]

Using Eshelby’s tensor \( S_{klmn} \) we can rewrite the last expression in the form of a set of six linear algebraic equations with respect to six unknown equivalent eigenstrains \( \epsilon^*_{mn} \):

\[
\left( (C^{\text{inh}}_{ijkl} - C) S_{klmn} + C_{ijmn} \right) \epsilon^*_{mn} = \left( C_{ijkl} - C^{\text{inh}}_{ijkl} \right) \epsilon^A_{kl} \tag{3.24}
\]

which allows us to determine the eigenstrain \( \epsilon^*_{mn} \) in the equivalent inclusion in terms of \( \epsilon^A_{kl} \).

In direct tensor rather than index notation the last expression has the form

\[
\left( (C^{\text{inh}} - C) S + C \right) \epsilon^* = -\left( C^{\text{inh}} - C \right) \epsilon^A. \tag{3.25}
\]

Since we determined the eigenstrain \( \epsilon^* \) in the equivalent inclusion in terms of the applied strain \( \epsilon^A \) via Eq. 3.25, the expression 3.18 for the strain inside the inhomogeneity can be rewritten as

\[
\epsilon^{\text{inh}} = \epsilon^A + \epsilon^c = \epsilon^A + S \epsilon^* = \epsilon^A - S \left( (C^{\text{inh}} - C) S + C \right)^{-1} (C^{\text{inh}} - C) \epsilon^A
\]

\[
= \left[ I - S \left( (C^{\text{inh}} - C) S + C \right)^{-1} (C^{\text{inh}} - C) \right] \epsilon^A. \tag{3.26}
\]

interaction tensor \( T \).
Tensor $T$ relating the strain inside the inhomogeneity $\epsilon^{(inh)}$ with the applied strain $\epsilon^A$ is called interaction tensor. It serves an important role in the Eshelby-based approaches.

additional task: determine the total elastic energy and enthalpy of the inhomogeneity problem

Eq. [3.23] can be simplified in case of an isotropic material. Consider Hook’s law in the form with Lamé constants $\lambda$ and $\mu$:

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}. \quad (3.27)$$

Eq.[3.19] for the bulk and deviatoric parts of the elastic stress field (considering the isotropic material behavior according to Eq. [3.27]) in the equivalent inclusion has the form

$$\sigma^I = 3k (\sigma^A + \sigma^c - \sigma^*), \quad \sigma^I_{ij} = 3\mu (\epsilon^A_{ij} + \epsilon^c_{ij} - \sigma^*). \quad (3.28)$$

Here

$$\epsilon_{ij} = \epsilon_{ij} + \frac{1}{3} \epsilon \delta_{ij}, \quad \epsilon = \epsilon_{ii}, \quad \epsilon_{ij} = \epsilon_{ij} - \frac{1}{3} \epsilon \delta_{ij}, \quad \sigma = 3k\epsilon, \quad \sigma_{ij} = 2\mu \epsilon_{ij}, \quad k = \lambda + 2/3\mu. \quad (3.29)$$

The backprime stands for the deviatoric values.

So Eq. [3.23] can be simplified to

$$k^{(inh)} (\epsilon^A + \epsilon^c) = k (\epsilon^A + \epsilon^c - \epsilon^*), \quad (3.30)$$

$$\mu^{(inh)} (\epsilon^A_{ij} + \epsilon^c_{ij}) = \mu (\epsilon^A_{ij} + \epsilon^c_{ij} - \sigma^*). \quad (3.31)$$

Using Eshelby’s tensor we can rewrite two last expressions in the form of a set of equations with respect to the unknown equivalent eigenstrains.

3.4. Example of a void under uniform external load - a step towards cracks

If we want to consider a void, the elastic stiffness tensor of the inhomogeneity has to be taken equal to zero $C^{(inh)}_{ijkl} = 0$. The solution for an ellipsoidal inhomogeneity under uniform loading is considered in section 3.3. In the case of a void the solution is further simplified since the stress inside the void is zero.

The equality [3.23] between the stresses in the inhomogeneity and the equivalent inclusion takes the form:

$$0 = C^{(inh)}_{ijkl} (\epsilon^A_{kl} + \epsilon^c_{kl}) = C_{ijkl} (\epsilon^A_{kl} + \epsilon^c_{kl} - \epsilon^*_{kl}). \quad (3.32)$$

From $C^{(inh)}_{ijkl} = 0$ follows that

$$\epsilon^A_{kl} + \epsilon^c_{kl} - \epsilon^*_l = 0 \quad (3.33)$$

or

$$-\epsilon^A_{kl} = \epsilon^c_{kl} - \epsilon^*_l. \quad (3.34)$$

Using Eq. 3.33, the total strain inside the inhomogeneity (void) is found to be equal to the eigenstrain of the equivalent inclusion:

$$\epsilon^I_{kl}^{(inh)} = \epsilon^A_{kl} + \epsilon^c_{kl} = \epsilon^*_l. \quad (3.35)$$
As it was mentioned earlier, the stresses inside the void are equal to zero, so should be the stresses in the equivalent inclusion (and this is seen by the last formula where the total strain $= \text{the eigenstrain}$).

From Eq. 3.34 we obtain the system of equations to determine the eigenstrain $\epsilon_{mn}^*$ in the equivalent inclusion in terms of applied stress $\sigma_{ij}^A$:

$$-\sigma_{ij}^A = C_{ijkl}(\epsilon_{kl}^c - \epsilon_{kl}^*) = C_{ijkl}(S_{klmn} - \delta_{km}\delta_{ln})\epsilon_{mn}^*.$$  \hspace{1cm} (3.36)

Additional observation: the last equation means that the applied stress must be canceled by the total stress inside the equivalent inclusion (when no external stress is applied to the system with the equivalent inclusion).

**additional task:** determine the extra enthalpy of the void
4. Effective moduli

N.B. During the derivation of formulas in this chapter the term "inclusion" stands for "inhomogeneity" according to notations in Chapter 3. It is due to the fact that in technical applications "inhomogeneity" is often called "inclusion", cf. "we have several inclusions in this matrix".

It is known that mixture of materials often provide advantageous properties. However, not all the properties can be simultaneously enhanced, some of them degrade. Only a rigorous discipline of heterogeneous material behavior can provide the key to optimizing material utilization.

In the following chapter the emphasis will be made on stiffness and strength of the heterogeneous material systems, since they are one of the most important engineering properties. By effective stiffness properties it is understood the average measure of the stiffness of the composite material, taking into account the properties of all phases of the heterogeneous media and their interaction. In some cases it is possible to obtain an exact solution for the effective moduli. When it is not possible, the upper and lower bounds on the effective moduli can be found.

The concept of equivalent homogeneity

On some sufficiently small scale (starting at the scale of atoms and molecules) all materials are heterogeneous. To be able to make engineering calculations, the continuum hypothesis (a statistical averaging process) is introduced and the material is taken as homogeneous.

In the current course we consider only conditions of heterogeneity with an abrupt change in properties across interfaces.

Scale of inhomogeneity

We assume the existence of a characteristic dimension of the inhomogeneity. For example, in a fiber system this could be the mean distance between fibers. The length scale of averaging \( \delta \) must be of a dimension much larger than that of the characteristic dimension of the inhomogeneity. A very advantageous circumstance occurs if there exists such a length scale of averaging \( \delta \) that is still small compared with the characteristic dimension of the body. Under this condition the material can be idealized as being effectively homogeneous.

In all further considerations we assume that the scale of properties averaging exists and is meaningful. That is, the scale of the inhomogeneity is assumed to be orders of magnitude smaller than the characteristic dimension of the problem of interest, such that there exists intermediate dimension over which the properties averaging can be legitimately performed. The condition just described is said to be that of effective or equivalent homogeneity (sometimes called macroscopic homogeneity and statistical homogeneity).

The basic problem for the heterogeneous material behavior is to utilize the averaging process to predict the effective properties of the idealized homogeneous medium in terms of the properties of the individual phases and some information on the interfacial geometry.

Volumetric averaging

First, we introduce a representative volume element (RVE) of the heterogeneous material having a characteristic dimension identical with that of the averaging dimension \( \delta \), defined above. It is shown schematically in Figs. 4.1 and 4.2. V stands for the volume of the RVE. Under conditions
Figure 4.1.: Representative volume element (RVE) for a matrix with inclusions and a void (left) and for a polycrystalline material (right).

Figure 4.2.: Representative volume element (RVE) for a periodic structure (left) and for an aligned fiber system (right).

of an imposed macroscopically homogeneous stress or deformation field on the RVE, the average
stress is defined as
\[
\langle \sigma_{ij} \rangle = \frac{1}{V} \int_V \sigma_{ij}(x_k) \text{d}v
\] (4.1)
and the average strain by
\[
\langle \epsilon_{ij} \rangle = \frac{1}{V} \int_V \epsilon_{ij}(x_k) \text{d}v.
\] (4.2)
Here \( \epsilon_{ij} \) is the infinitesimal strain tensor.

It is important to notice, that Eqs. 4.1 and 4.2 are very general in the sense that no restrictions whatsoever are placed on the interfacial geometry of the heterogeneous combination of phases. The effective linear stiffnesses designated by the tensor \( C_{ijkl} \) are defined through their presence in the relation
\[
\langle \sigma_{ij} \rangle = C_{ijkl} \langle \epsilon_{kl} \rangle.
\] (4.3)

To solve for the effective properties of the heterogeneous media, we need to perform the averaging process specified by Eqs. 4.1 and 4.2 and thence solve for \( C_{ijkl} \) from Eq. 4.3.

The idea is quite simple, but the realization in many cases is quite cumbersome.

We will now derive an explicit formula that will be used to obtain the effective properties in the heterogeneous media. Consider a two-material heterogeneous system, one material of which is continuous and the other in the form of discrete inclusions. Both material are taken to be isotropic.

The stress-strain relations are given by
\[
\sigma_{ij} = \lambda_1 \epsilon_{kk} \delta_{ij} + 2\mu_1 \epsilon_{ij}
\] (4.4)
for the inclusion phase and
\[
\sigma_{ij} = \lambda_M \epsilon_{kk} \delta_{ij} + 2\mu_M \epsilon_{ij}
\] (4.5)
for the matrix phase.

The average stress formula 4.1 can be written as
\[
\langle \sigma_{ij} \rangle = \frac{1}{V} \int_V \sum_{n=1}^N \sigma_{ij} \text{d}v + \frac{1}{V} \sum_{n=1}^N \int_{V_n} \sigma_{ij} \text{d}v.
\] (4.6)
Here \( N \) inclusions with volumes \( V_n \) within the RVE are taken into account.

Using the stress-strain relationship for the matrix phase, the last equation can be expressed as
\[
\langle \sigma_{ij} \rangle = \frac{1}{V} \int_{V-\sum_{n=1}^N V_n} (\lambda_M \epsilon_{kk} \delta_{ij} + 2\mu_M \epsilon_{ij}) \text{d}v + \frac{1}{V} \sum_{n=1}^N \int_{V_n} \sigma_{ij} \text{d}v.
\] (4.7)

The first integral can be decomposed into two integrals, so
\[
\langle \sigma_{ij} \rangle = \frac{1}{V} \int_V (\lambda_M \epsilon_{kk} \delta_{ij} + 2\mu_M \epsilon_{ij}) \text{d}v - \frac{1}{V} \sum_{n=1}^N \int_{V_n} (\lambda_M \epsilon_{kk} \delta_{ij} + 2\mu_M \epsilon_{ij}) \text{d}v + \frac{1}{V} \sum_{n=1}^N \int_{V_n} \sigma_{ij} \text{d}v.
\] (4.8)

1Similar volume averaging definitions apply in the nonlinear case.
The average stress term can be rewritten with the use of Eq. 4.3 and the first integral can be explicitly written in terms of average strains:

\[ C_{ijkl} \langle \epsilon_{kl} \rangle = \lambda_M \langle \epsilon_{kk} \rangle \delta_{ij} + 2\mu_M \langle \epsilon_{ij} \rangle + \frac{1}{V} \sum_{n=1}^{N} \int_{V_n} (\sigma_{ij} - \lambda_M \epsilon_{kk} \delta_{ij} - 2\mu_M \epsilon_{ij}) \, dv. \]  (4.9)

From the last formula follows, that only the conditions within the inclusions are needed for the evaluation of the effective properties tensor \( C_{ijkl} \).

**Dilute suspensions**

Dilute suspension corresponds to the behavior of a single inclusion in an infinite matrix phase. The physical meaning is that the particles are small and so far away of each other that all interactions between particles can be neglected.

Consider the dilute suspension in a state of imposed simple shear deformation \( \epsilon_{12} \) at large distances from the inclusion. In this case formula 4.9 takes the form

\[ 2\mu \langle \epsilon_{ij} \rangle = 2\mu_M \langle \epsilon_{ij} \rangle + \frac{1}{V} \int_{V_1} (2\mu_1 \epsilon_{12} - 2\mu_M \epsilon_{12}) \, dv. \]  (4.10)

With \( \epsilon_{12} = \langle \epsilon \rangle \) and taking into account that a single ellipsoidal inclusion embedded in an infinite medium is in a state of homogeneous deformation (corresponding to that imposed in the infinite media), the last formula can be written as

\[ \mu \langle \epsilon \rangle = \mu_M \langle \epsilon \rangle + \frac{V_1}{V} (\mu_1 - \mu_M) \langle \epsilon_1 \rangle. \]  (4.11)

Here \( \langle \epsilon_1 \rangle \) is the uniform shear strain in the inclusion.

The formula 4.11 can be rewritten as

\[ \frac{\mu - \mu_M}{\mu_1 - \mu_M} = \frac{\langle \epsilon_1 \rangle}{\langle \epsilon \rangle}, \]  (4.12)

where \( c \) is the volume fraction of the inclusions

\[ c = \frac{V_1}{V}. \]  (4.13)

In a similar manner the effective bulk modulus \( k \) can be derived as

\[ \frac{k - k_M}{k_1 - k_M} = \frac{\langle \epsilon_1 \rangle}{\langle \epsilon \rangle}, \]  (4.14)

where \( \langle \epsilon \rangle \) is the average dilatational strain and \( \langle \epsilon_1 \rangle \) is the uniform dilatational strain in the inclusion.

The formulas 4.12 and 4.14 show that it is only necessary to know the state of uniform strain in the inclusion in order to derive the effective properties.

The results 4.12 and 4.14 are valid only for dilute suspension conditions, whereas the formula 4.9 is a general one.
Energy methods

There is an alternate method to the explicit use of formula 4.3 for defining the effective properties. It is realized through energy equivalence. Contract expression 4.3 with the average strain tensor \( \langle \epsilon_{ij} \rangle \) to obtain

\[
\langle \sigma_{ij} \rangle \langle \epsilon_{ij} \rangle = C_{ijkl} \langle \epsilon_{ij} \rangle \langle \epsilon_{kl} \rangle.
\]

(4.15)

The average stress and strain terms in formula 4.15 are due to the imposed conditions on the boundary of the RVE. Since the states of stress and deformation are macroscopically homogeneous within the RVE, the averages can be obtained equivalently from boundary values and through volume integrations. Consequently, the work term on the left-hand side of Eq. 4.15 is equivalent to the work integral calculated from surface stresses and displacements, so

\[
\int_S \sigma_i u_i \, ds = C_{ijkl} \langle \epsilon_{ij} \rangle \langle \epsilon_{kl} \rangle.
\]

(4.16)

Using divergence theorem and the fact that \( \sigma_{ij,j} = 0 \), Eq. 4.16 get the form

\[
\frac{1}{2} \int_V \sigma_{ij} \epsilon_{ij} \, dv = \frac{1}{2} C_{ijkl} \langle \epsilon_{ij} \rangle \langle \epsilon_{kl} \rangle.
\]

(4.17)

Formula 4.17 defines the effective properties \( C_{ijkl} \) through the equality of the strain energy stored in the heterogeneous media to that stored in the equivalent homogeneous media. Conducting some transformation to Eq. 4.9 one gets the following energy form

\[
\frac{1}{2} C_{ijkl} \langle \epsilon_{ij} \rangle \langle \epsilon_{kl} \rangle = \frac{U_0}{V} + \frac{1}{2} \sum_{n=1}^{N} \int_{V_n} (\sigma_{ij} \epsilon_{ij}^0 - \sigma_{ij}^0 \epsilon_{ij}) \, dv,
\]

(4.18)

where \( U_0 \) is a strain energy corresponding to the medium composed entirely of matrix material and \( \sigma_{ij}^0 \) are the stresses in the inclusions under the condition that the inclusions have been replaced by the matrix material.

additional task: conduct the derivation of formula 4.18 from 4.9

If now use formula 4.17 in the left-hand side of Eq. 4.18 and apply the divergence theorem to the integral terms we come to the expression identical to Eshelby’s formula, generalized to the case of \( N \) inclusions:

\[
U = U_0 + \frac{1}{2} \int_{S_i} (\sigma_i u_i^0 - \sigma_i^0 u_i) \, ds.
\]

(4.19)

Thus, formula 4.18 is the appropriate and operational energy form to be used to solve for the effective properties \( C_{ijkl} \). Only the solution for the stress and strain fields inside the inclusions are needed for the evaluation of formula 4.18.

It should be noticed that Eshelby’s formula is more general than the specific formula 4.18.
4.1. Spherical inclusions

4.1.1. Dilute suspension (power series solution)

Shear modulus

In this section we derive the solution for the effective shear modulus of a dilute suspension of elastic spherical particles in a continuous phase of another elastic material.

Consider first the case of a homogeneous medium in a state of pure shear deformation (cf. Fig. 4.3). With reference to an \((x_1, x_2, x_3)\) rectangular Cartesian CS, the displacement components are specified as

\[
\begin{align*}
    u_1 &= qx_1, \\
    u_2 &= -qx_2, \\
    u_3 &= 0,
\end{align*}
\]

Figure 4.3.: Applied in-plane loading and deformation state at the material point of the body.

We convert to a spherical CS \((r, \theta, \phi)\) (Fig. 4.4) so the applied displacements 4.20 are expressed as

\[
\begin{align*}
    u_r &= qr \sin^2 \theta \cos 2\phi, \\
    u_\theta &= qr \sin \theta \cos \theta \cos 2\phi, \\
    u_\phi &= -qr \sin \theta \sin 2\phi.
\end{align*}
\]

Please check the values of \(u_r, u_\theta, u_\phi\) for different values of the angles \(\theta\) and \(\phi\) and compare them with the displacements prescribed by formula 4.20.

Guided by the preceding forms for homogeneous media deformation, a general solution for the heterogeneous problem is assumed in the form

\[
\begin{align*}
    u_r &= U_r(r) \sin^2 \theta \cos 2\phi, \\
    u_\theta &= U_\theta(r) \sin \theta \cos \theta \cos 2\phi, \\
    u_\phi &= U_\phi(r) \sin \theta \sin 2\phi,
\end{align*}
\]
where \(U_r(r), U_\theta(r), U_\phi(r)\) are unknown functions of \(r\) to be solved from the equilibrium equations.

The equilibrium equations in spherical coordinates take the form

\[
\begin{align*}
(\lambda + \mu) \frac{\partial}{\partial r} K + \mu \left( \nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta \cot \theta}{\partial r} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right) &= 0, \\
(\lambda + \mu) \frac{1}{r} \frac{\partial}{\partial \theta} K + \mu \left( \nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\phi}{\partial \phi} \right) &= 0, \\
(\lambda + \mu) \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} K + \mu \left( \nabla^2 u_\phi - \frac{u_\phi}{r^2 \sin^2 \theta} + \frac{2}{r^2 \sin^2 \phi} \frac{\partial u_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \phi} \right) &= 0,
\end{align*}
\]  

(4.23, 4.24, 4.25)

where

\[
K = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{1}{r} \frac{\partial u_\phi}{\partial \phi}
\]

(4.26)

and Laplacian in Spherical CS is defined by Eq. D.1.

Substituting Eq. 4.22 in equilibrium Eqs. 4.23 - 4.25 and equating to zero the coefficients of \(\sin^2 \theta\)
and the terms independent of $\theta$ gives three governing equations

$$\begin{align*}
2(1 - \nu) \left( U''_r + \frac{2}{r} U'_r - \frac{2}{r^2} U_r - \frac{3}{r^2} U'_\theta + \frac{3}{r^2} U_\theta \right) + (1 - 2\nu) \left( -\frac{6}{r^2} U_r + \frac{3}{r^2} U'_\theta + \frac{3}{r^2} U_\theta \right) &= 0, \quad (4.27) \\
2(1 - \nu) \left( \frac{2}{r} U'_r + \frac{4}{r^2} U_r - \frac{6}{r^2} U_\theta \right) + (1 - 2\nu) \left( -\frac{2}{r} U'_r + U''_\theta + \frac{2}{r} U'_\theta \right) &= 0, \quad (4.28) \\
U_\theta + U_\phi &= 0, \quad (4.29)
\end{align*}$$

where the prime designates derivatives with respect to $r$. The solution of system of Eqs. 4.27 - 4.29 may be shown to be given by

$$\begin{align*}
U_r &= C_1 r - \frac{6\nu}{1 - 2\nu} C_2 r^3 + \frac{3C_3}{r^4} + \frac{5 - 4\nu}{1 - 2\nu} C_4, \quad (4.30) \\
U_\theta &= C_1 r - \frac{7 - 4\nu}{1 - 2\nu} C_2 r^3 - \frac{2C_3}{r^4} + \frac{2C_4}{r^2}, \quad (4.31) \\
U_\phi &= -U_\theta. \quad (4.32)
\end{align*}$$

Now, solutions of the form 4.30 - 4.31 are taken for the inclusion phase and the matrix phase:

$$\begin{align*}
U_{r1} &= A_1 r - \frac{6\nu}{1 - 2\nu} A_2 r^3, \quad (4.33) \\
U_{\theta 1} &= A_1 r - \frac{7 - 4\nu}{1 - 2\nu} A_2 r^3, \quad (4.34)
\end{align*}$$

and

$$\begin{align*}
U_{rM} &= B_1 r + \frac{3B_3}{r^4} + \frac{5 - 4\nu}{1 - 2\nu} B_4, \quad (4.35) \\
U_{\theta M} &= B_1 r - \frac{2B_3}{r^4} + \frac{2B_4}{r^2}. \quad (4.36)
\end{align*}$$

The particular terms in Eqs. 4.30 - 4.31 that are missing in Eqs. 4.33 - 4.36 are taken with vanishing coefficients to avoid unbounded or singular conditions. Coefficient $B_1$ is considered as being a given quantity ($B_1 = q$) since it specifies the state of imposed pure shear deformation at $r \to \infty$. Thus within Eqs. 4.33 - 4.36 there are four constants to be determined.

The interface conditions require the continuity of stresses $\sigma_{rr}, \sigma_{r\theta}, \sigma_{r\phi}$ and displacements $u_r, u_\theta, u_\phi$ at the interface $r = a$, where $a$ is the radius of the spherical inclusion. However, only four of these conditions are independent. The resulting continuity conditions across the inclusion-matrix interface become

$$\begin{align*}
aA_1 - \frac{6\nu}{1 - 2\nu} a^3 A_2 &= aB_1 + \frac{3}{a^3} B_3 + \frac{5 - 4\nu}{1 - 2\nu} \frac{B_4}{a^2}, \quad (4.37) \\
aA_1 - \frac{7 - 4\nu}{1 - 2\nu} a^3 A_2 &= aB_1 - \frac{2}{a^3} B_3 + \frac{2B_4}{a^2}, \quad (4.38) \\
21\lambda a^2 A_2 + 2\mu_1 \left( A_1 - \frac{18\nu}{1 - 2\nu} a^2 A_2 \right) &= -6\lambda a^3 B_4 + 2\mu_1 \left( B_1 - \frac{12}{a^3} B_3 - \frac{5 - 4\nu}{1 - 2\nu} \frac{B_4}{a^2} \right), \quad (4.39) \\
\mu_1 \left( A_1 - \frac{7 + 4\nu}{1 - 2\nu} a^2 A_2 \right) &= \mu_M \left( B_1 + \frac{8}{a^3} B_3 + \frac{1 + \nu}{1 - 2\nu} \frac{B_4}{a^2} \right). \quad (4.40)
\end{align*}$$
Now we use Eshelby’s formula \(4.19\) and the criterion from the preceding subsection that the equivalent homogeneous medium store the same level of strain energy as the suspension. In spherical CS Eshelby’s formula takes the form

\[
U = U_0 + \frac{1}{2} \int_{S_1} (\sigma_{rr}u_r^0 + \sigma_{\theta\theta}u_{\theta}^0 + \sigma_{\phi\phi}u_{\phi}^0 - \sigma_{rr}^0 u_r - \sigma_{\theta\theta}^0 u_{\theta} - \sigma_{\phi\phi}^0 u_{\phi}) \, ds. \tag{4.41}
\]

For the homogeneous problem (with superscript \(^0\)) consisting entirely of the matrix phase

\[
\begin{align*}
    u_r^0 &= B_1 r \sin^2 \theta \cos 2\theta, \tag{4.42} \\
    u_\theta^0 &= B_1 r \sin \theta \cos \theta \cos 2\phi, \tag{4.43} \\
    u_\phi^0 &= -B_1 r \sin \theta \sin 2\phi, \tag{4.44}
\end{align*}
\]

and

\[
\begin{align*}
    \sigma_{rr}^0 &= 2\mu_M B_1 \sin^2 \theta \cos 2\theta, \tag{4.45} \\
    \sigma_{\theta\theta}^0 &= 2\mu_M B_1 \sin \theta \cos \theta \cos 2\phi, \tag{4.46} \\
    \sigma_{\phi\phi}^0 &= -2\mu_M B_1 \sin \theta \sin 2\phi. \tag{4.47}
\end{align*}
\]

The strain energy stored in the homogeneous sphere of matrix material is

\[
U_0 = \frac{4}{3}\pi b^3 \mu_M B_1^2, \tag{4.48}
\]

where \(b\) is the radius of the sphere.

The strain energy stored in the equivalent homogeneous material is

\[
U_{\text{eq.homog.}} = \frac{4}{3}\pi b^3 \mu B_1^2, \tag{4.49}
\]

where \(\mu\) is the effective shear modulus.

Since

\[
U = U_{\text{eq.homog.}}, \tag{4.50}
\]

Eq. \(4.41\) results in

\[
\mu = \mu_M + \frac{3}{8\pi b^3 B_1^2} \int_0^{2\pi} \int_0^\pi (\sigma_{rr}u_r + \sigma_{\theta\theta}u_{\theta} + \sigma_{\phi\phi}u_{\phi} - \sigma_{rr}^0 u_r - \sigma_{\theta\theta}^0 u_{\theta} - \sigma_{\phi\phi}^0 u_{\phi}) a^2 \sin \theta d\theta d\phi. \tag{4.51}
\]

Here the stresses and displacements are those at the interface \((r = a)\). The terms with \(^0\) superscript are taken from Eqs. \(4.42\) - \(4.47\). The other terms are the solutions of the suspension problem: displacement are defined by Eqs. \(4.22\) taking into account \(4.33\) - \(4.34\) or \(4.35\) - \(4.36\). Strain-displacement relationships in the spherical coordinates have the form \(C\). The stresses follow directly from the linear elasticity stress-strain relations - add these relations. The stresses and displacements may be evaluated in either phase at the interface.

Performing the integration in the last equation and using the dilute suspension condition \((a/b)^3 \ll 1\), we come to the following formula:

\[
\frac{\mu}{\mu_M} = 1 - \frac{15(1 - \nu_M) \left(1 - \frac{\mu}{\mu_M}\right) c}{7 - 5\nu_M + 2(4 - 5\nu_M) \frac{\mu}{\mu_M}}, \tag{4.52}
\]

38
where \( c = (a/b)^3 \) - the volume fraction of the spherical particles under dilute conditions.

It is interesting to examine the solution in the special case where the inclusion is perfectly rigid and the matrix material is incompressible. The result is

\[
\frac{\mu}{\mu_M} = 1 - \frac{5}{2}c. \tag{4.53}
\]

**Bulk modulus**

The procedure used above to determine effective shear modulus for a dilute elastic suspension can also be followed to determine the effective bulk modulus (this is simpler due to spherical symmetry ⇒ one-dimensionality).

The final formula is

\[
k = k_M + \frac{(k_1 - k_M)c}{1 + (k_1 - k_M)/(k_M + \frac{4}{3}\mu_M)}. \tag{4.54}
\]

### 4.1.2. Dilute suspension (Eshelby solution)

As an alternative to the power series solution (subsection 4.1.1), the Eshelby solution can be used for the dilute concentration of spheres problem to find the stress field in the spherical inhomogeneity.

**Bulk Modulus**

From Eshelby’s equivalence principle (cf. 3.3 Eqs. 3.23, 3.30):

\[
k_1(\epsilon^A + \epsilon^c) = k_M(\epsilon^A + \epsilon^c - \epsilon^*), \tag{4.55}
\]

where

\[
\epsilon^A = \epsilon^A_{ii} = \epsilon^A_{rr} + \epsilon^A_{\theta\theta} + \epsilon^A_{\phi\phi} = 3B_1
\]

according to Eq. 3.29 and boundary condition. Constrained strain is defined as:

\[
\epsilon_{ij}^c = S_{ijkl}\epsilon_{kl}^*. \tag{4.56}
\]

In case of hydrostatic loading

\[
\epsilon_{ii}^c = \epsilon^c = S_{ijkl}\epsilon_{kl}^* = \frac{1}{3}S_{iikk}\epsilon^*, \tag{4.57}
\]

so

\[
\epsilon^c = \frac{1}{3}(S_{1111} + S_{1122} + S_{1133} + S_{2211} + S_{2222} + S_{2233} + S_{3311} + S_{3322} + S_{3333})\epsilon^* =
\]

\[
= (S_{1111} + 2S_{1122})c^* E_{2.33} \left[ 7 - 5\nu_M \right] + 2 \frac{5\nu_M - 1}{15(1 - \nu_M)} \left[ 15(1 - \nu_M) \right] \epsilon^* = \frac{5 + 5\nu_M}{15(1 - \nu_M)} \epsilon^*. \tag{4.58}
\]

Finally, bulk constraint strain is expressed via bulk eigenstrain as

\[
\epsilon^c = \frac{1 + \nu_M}{3(1 - \nu_M)} \epsilon^*. \tag{4.59}
\]
Substituting Eqs. 4.56 and 4.59 in Eq. 4.55, we get
\[ k_1 \left( 3B_1 + \frac{1 + \nu_M}{3(1 - \nu_M)} \epsilon^* \right) = k_M \left( 3B_1 + \frac{1 + \nu_M}{3(1 - \nu_M)} \epsilon^* - \epsilon^* \right). \] (4.60)

Conducting basic algebraic transformations of Eq. 4.60
\[ 3B_1(k_M - k_1) = \left[ k_1 \frac{1 + \nu_M}{3(1 - \nu_M)} + k_M \frac{3(1 - \nu_M) - (1 + \nu_M)}{3(1 - \nu_M)} \right] \epsilon^* = \frac{k_1(1 + \nu_M) + 2k_M(1 - 2\nu_M)}{3(1 - \nu_M)} \epsilon^* \]
leads to the following expression for the equivalent eigenstrain:
\[ \epsilon^* = \frac{9B_1(k_M - k_1)(1 - \nu_M)}{k_1(1 + \nu_M) + 2k_M(1 - 2\nu_M)}. \] (4.61)

From Eq. 4.55 follows that
\[ \epsilon^A + \epsilon^c = \frac{k_M}{k_M - k_1} \epsilon^* \text{ Eqs. 4.62} \frac{9B_1k_M(1 - \nu_M)}{k_1(1 + \nu_M) + 2k_M(1 - 2\nu_M)}. \] (4.63)

Relation between the bulk stress and strain has the form
\[ \sigma = 3k_1(\epsilon^A + \epsilon^c) \text{ Eqs. 4.63} \frac{27B_1k_1k_M(1 - \nu_M)}{k_1(1 + \nu_M) + 2k_M(1 - 2\nu_M)}. \] (4.64)

or, taking into account relations between the elastic constants specified in Appendix A
\[ \sigma = \frac{27B_1k_1k_M(1 - \nu_M)}{k_1(1 + \nu_M) + 2k_M(1 - 2\nu_M)} = \frac{27B_1k_1k_M \left( 1 - \frac{3k_M - 2\mu_M}{2(3k_M + \mu_M)} \right)}{k_1 \left( 1 + \frac{3k_M - 2\mu_M}{2(3k_M + \mu_M)} \right) + 2k_M \left( 1 - \frac{3k_M - 2\mu_M}{2(3k_M + \mu_M)} \right)} = \]
\[ = \frac{27B_1k_1k_M[6k_M + 2\mu_M - 3k_M + 2\mu_M]}{k_1[6k_M + 2\mu_M + 3k_M - 2\mu_M]} = \frac{9B_1k_1k_M \left( 4\mu_M + 3k_M \right)}{6k_M + 2\mu_M + 3k_M - 2\mu_M}. \]
(4.65)

the above equation is similar to the result obtained by power series method.

4.1.3. Mori-Tanaka model

4.1.4. Composite spheres model

In the previous subsections 4.1.1 and 4.1.2 we determined the effective bulk and shear moduli for the case of dilute suspension of spherical particles.

Consider now the nondilute case. We can introduce various geometrical models to resemble the real particles distribution and than try to find the homogenized properties.

The composite spheres model was introduced by Hashin in 1962. In this model a spherical particles of different sizes fill the matrix, see Fig. 4.5.

The dash curves with radius \( b \) correspond to a matrix phase associated with each particle of radius \( a \) (solid curves). It is important to notice that ratio \( a/b \) is taken constant for each composite
sphere. Since it is necessary to fill all the volume, we have also to consider the distribution size down to infinitesimal. So this model is good for composites with fine gradation of sizes of spherical inclusions. Conversely, this model is not suitable for composites with single size particles at high concentrations. At the end of the current subsection we conclude about the general applicability of this model.

**Bulk modulus**

We begin with an analysis of a single composite sphere and generalize this result to the RVE. Let us subject the composite sphere to a hydrostatic stress \( p \) on its outer boundary:

\[
\sigma_{rr} = p \quad \text{at} \quad r = b.
\]  

(4.66)

And an equivalent homogeneous material (equivalent homogeneous sphere) is subjected to the same stress.

The displacements at the outer boundary of the composite sphere are equated with the displacements of the equivalent homogeneous sphere to provide the same average state of dilatation. By this procedure we can determine the effective bulk modulus of the single composite sphere. In case of the applied hydrostatic stress \( p \) to the single composite sphere it is necessary to satisfy the single equilibrium equation:

\[
\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r} (\sigma_{rr} - \sigma_{\theta \theta}) = 0.
\]

(4.67)

It is important not to forget that for the discussed case \( \sigma_{\phi \phi} = \sigma_{\theta \theta} \).

Taking into account a linear constitutive relationship for the isotropic homogeneous material and
the following kinematic equations

\[
\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad (4.68) \\
\epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad (4.69)
\]

we can write Eq. 4.67 in terms of displacements as

\[
\frac{\partial^2 u_r}{\partial r^2} + \frac{2}{r} \frac{\partial u_r}{\partial r} - \frac{2}{r^2} u_r = 0. \quad (4.70)
\]

The solution of the last equation is given by

\[
u_r = Ar + \frac{B}{r^2}. \quad (4.71)
\]

Adopting this solution for inclusion and matrix gives

\[
u_{rI} = A_I r, \quad (4.72) \\
u_{rM} = A_M r + \frac{B_M}{r^2}. \quad (4.73)
\]

Similar to subsection 4.1.1, \(B_I\) has be zero to avoid the singularity.

The radial stresses are calculated according to the formula

\[
\sigma_{rr} = \lambda (\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}) + 2\mu \epsilon_{rr}, \quad (4.74)
\]

where

\[
\epsilon_{rrI} = \epsilon_{\phi\phiI} = \epsilon_{\theta\thetaI} = A_I, \quad (4.75) \\
\epsilon_{rrM} = A_M - 2 \frac{B_M}{r^3}, \quad \epsilon_{\phi\phiM} = \epsilon_{\theta\thetaM} = A_M + \frac{B_M}{r^3}. \quad (4.76)
\]

Substituting the strains in the inclusion (formula 4.75) (notice that, as expected, strain in inclusion is constant) and in the matrix (formula 4.76) into formula 4.74, we derive expressions for the radial stresses in the inclusion and the matrix:

\[
\sigma_{rrI} = (3\lambda_I + 2\mu_I)A_I, \quad (4.77) \\
\sigma_{rrM} = (3\lambda_M + 2\mu_M)A_M - 4\mu_M \frac{B_M}{r^3}. \quad (4.78)
\]

Three unknown constants are defined from the boundary condition 4.66 (compare this BC with the one for the dilute distribution of the spherical inclusions) and continuity conditions

\[
\sigma_{rrI} = \sigma_{rrM} \text{ at } r = a, \quad (4.79) \\
u_{rI} = u_{rM} \text{ at } r = a. \quad (4.80)
\]
After simple calculations the constants are found as

\[ A_M = L \frac{p}{4\mu_M}, \quad (4.81) \]
\[ B_M b^3 = \frac{p}{4\mu_M} \left( -1 + \frac{3k_M}{4\mu_M} L \right), \quad (4.82) \]
\[ A_I = A_M + \frac{B_M}{a^3}. \quad (4.83) \]

Here

\[ L = \frac{3k_I + 4\mu_M}{3(k_I - k_M)c + \frac{4}{3}k_k k_M/\mu_M + 3k_M}. \quad (4.84) \]

The displacement at the outer boundary of the composite sphere:

\[ u_{rM} |_{r=b} = A_M b + \frac{B_M}{b^2}. \quad (4.85) \]

The displacement at the boundary of the equivalent homogeneous sphere:

\[ u_r |_{r=b} = \frac{pb}{3k}, \quad (4.86) \]

where \( \hat{k} \) is the effective bulk modulus of the equivalent homogeneous sphere.

Equating Eqs. \(4.85\) and \(4.85\) leads to

\[ A_M b + \frac{B_M}{b^2} = \frac{pb}{3k} \quad (4.87) \]

or

\[ L + (-4\mu_M + 3k_M L) = \frac{4\mu_M}{3k} \quad (4.88) \]

and after further transformations the effective bulk modulus of the equivalent homogeneous sphere is determined as

\[ \hat{k} = k_M + \frac{c(k_I - k_M)}{1 + (1 - c) \frac{k_k - k_M}{k_M + 3\mu_M}}. \quad (4.89) \]

Now comes the important part of the derivations where it is important to show that the solution for the effective bulk modulus \(4.89\) of the simple composite sphere can be applied to the entire RVE.

To start with, the same pressure \( p \) could have been applied to all composite spheres in the RVE. From this follows that the stress state satisfies the equations of equilibrium and is continuous throughout the entire RVE. From the point of view of the theorem of minimum complementary energy such stress state is an admissible stress state and the corresponding energy is an upper bound to that of the actual stress state in the RVE.

For an isotropic material in a state of dilatation the local complementary energy is defined as

\[ U = \frac{p^2}{2k} \quad (4.90) \]
and it follows that
\[
\frac{1}{k} \leq \frac{1}{\tilde{k}}.
\]
(4.91)

Here \( k \) is the effective bulk modulus. So we found a lower bound for the effective bulk modulus.

Now impose a displacement BC on the single composite sphere and solve for an effective bulk modulus \( \tilde{k} \) of the composite sphere by requiring that the average strain state is equal in the composite sphere and in the equivalent homogeneous sphere. In sense of the theorem of minimum potential energy the displacement field for the single composite sphere can be viewed as an admissible displacement field for the RVE (we can as well add a rigid body movement component to each sphere). It now follows that the effective bulk modulus \( \tilde{k} \) of the single composite sphere is an upper bound to that of the RVE:
\[
k \leq \tilde{k}.
\]
(4.92)

It was found that \( \tilde{k} = \tilde{k} \), thus the bounds for the effective bulk modulus coincide. The effective bulk modulus is given by formula (4.89) which can be rewritten as
\[
\frac{k - k_M}{k_I - k_M} = \frac{c}{1 + (1 - c) \frac{k_I - k_M}{k_M + \frac{4}{3} \mu_M}}.
\]
(4.93)

Shear modulus

The problem of determining the effective shear modulus \( \mu \) is expected to be more complicated that that for \( k \) because it is a 3D elasticity problem in contrast to a 1D elasticity problem for \( k \). We can follow the procedure similar to that for determining the effective bulk modulus and use minimum theorems to obtain upper and lower bounds on \( \mu \). It was found that this bound do not coincide, except for a very small (corresponds to the dilute solution case) or very large volume concentrations. The large volume concentration result is found as (valid to the first order of \( c' = 1 - c \))
\[
\frac{\mu}{\mu_I} = 1 - \frac{(1 - \frac{\mu}{\mu_M}) \left(7 - 5\nu_M + 2(4 - 5\nu_M) \frac{\mu}{\mu_M}\right)}{15(1 - \nu_M)} c'.
\]
(4.94)

When simple shear type displacement components are prescribed on the surface of the composite sphere, it is found that the resulting boundary stresses do not correspond to a state of simple shear stress. Or, reversely, when the simple shear stresses are prescribed on the boundary, the resulting surface deformation state does not fully correspond to that of simple shear deformation. So the fact that the upper and lower bounds do not coincide in the shear problem is not surprising. It was found that the bigger the discrepancy between \( \mu_I \) and \( \mu_M \), the greater is the difference between the bonds. It follows that it is necessary to elaborate a different approach to find the effective shear modulus for the composite spheres model.

4.1.5. A three-phase model (generalized self-consistent method)

As an attempt to determine the effective shear modulus for the composite spheres model another approach is proposed. Consider only one composite sphere taken from the composite spheres model and surround it by the matrix with equivalent homogeneous properties (Fig. 4.6). In such a way we replaced the all other composite spheres with the equivalent homogeneous matrix with the unknown
Figure 4.6.: Illustration of the three-phase model: orange - spherical inclusion, green - matrix, blue shaded - equivalent homogeneous medium.

effective properties \( \mu \) and \( k \). Suppose that the composite system is under (applied) homogeneous deformations at infinity. The composite system is equal to a completely homogeneous material when they store equal amount of the strain energy under conditions of identical average strain.

In contrast to subsection 4.1.4, the effective properties are now included in both problems of composite material (because of the properties of the outer matrix) and equivalent homogeneous media.

We can assume that the effective properties obtained from the solution of the three-phase model correspond to the ones from the composite spheres model. However, it is a hypothesis which can be confirmed or rejected after the solution of the current problem is derived. It was found that the effective bulk modulus \( k \) is the same in both models after conducting all necessary derivations.

Now proceed to the calculation of the effective shear modulus of the three-phase model. Nevertheless, it is an open question whether both models are identical in terms of solution for \( \mu \). Even it is not, the results for the three-phase model are of interest in its own.

The solution of the problem closely follows the one for the dilute suspension model. A general solution for the heterogeneous problem under the remote simple shear deformation is also assumed in the form 4.22. From the equilibrium equations, the functions \( U_r(r) \), \( U_\theta(r) \), \( U_\phi(r) \) are found in the form 4.30 - 4.32. In a similar manner as in subsection 4.1.1 we write a separate solution for each of three regions:

\[
U_r = A_1 r - \frac{6\nu_1}{1-2\nu_1} A_2 r^3, \quad (4.95)
\]

\[
U_\theta = A_1 r - \frac{7-4\nu_1}{1-2\nu_1} A_2 r^3, \quad (4.96)
\]
\[ U_{rM} = B_1 r - \frac{6\nu_M}{1 - 2\nu_M} B_2 r^3 + \frac{3B_3}{r^2} + \frac{5 - 4\nu_M}{1 - 2\nu_M} B_4, \]
\[ U_{\theta M} = B_1 r - \frac{7 - 4\nu_M}{1 - 2\nu_M} B_2 r^3 - \frac{2B_3}{r^2} + \frac{2B_4}{r^2}, \]
\[ U_{rEQ} = D_1 r + \frac{3D_3}{r^2} + \frac{5 - 4\nu}{1 - 2\nu} D_4, \]
\[ U_{\theta EQ} = D_1 r - \frac{2D_3}{r^2} + \frac{2D_4}{r^2}. \]

Here "EQ" stands for the equivalent homogeneous media and \( U_\phi = -U_\theta \) in every region. Constant \( D_1 \) is given by the imposed state of simple shear at large distance from the composite sphere. There are eight constants to be determined from four stress and displacement continuity conditions at \( r = a \) and four stress and displacement continuity conditions at \( r = b \) (similar to subsection 4.1.1).

\[ aA_1 - \frac{6\nu_i}{1 - 2\nu_i} a^3 A_2 = aB_1 - \frac{6\nu_M}{1 - 2\nu_M} a^3 B_2 + \frac{3B_3}{a^2} + \frac{5 - 4\nu_M}{1 - 2\nu_M} B_4, \]
\[ aA_1 - \frac{7 - 4\nu_M}{1 - 2\nu_M} a^3 A_2 = aB_1 - \frac{7 - 4\nu_M}{1 - 2\nu_M} a^3 B_2 - \frac{2B_3}{a^2} + \frac{2B_4}{a^2}, \]
\[ 21\lambda_1 a^2 A_2 + 2\mu_1 \left( A_1 - \frac{18\nu_i}{1 - 2\nu_i} a^2 A_2 \right) \]
\[ = \lambda_M \left( 21a^2 B_2 - \frac{6}{a^3} B_4 \right) + 2\mu_M \left( B_1 - \frac{18\nu_M}{1 - 2\nu_M} a^2 B_2 - \frac{12}{a^3} B_3 - \frac{5 - 4\nu_M}{1 - 2\nu_M} B_4 \right), \]
\[ \mu_1 \left( A_1 - \frac{7 + 2\nu_i}{1 - 2\nu_i} a^2 A_2 \right) = \mu_M \left( B_1 - \frac{7 + 2\nu_M}{1 - 2\nu_M} a^2 B_2 + \frac{8}{a^3} B_3 + \frac{2\nu_1}{1 - 2\nu} B_4 \right). \]

and

\[ bB_1 - \frac{6\nu_M}{1 - 2\nu_M} b^3 B_2 + \frac{3B_3}{b^4} + \frac{5 - 4\nu_M}{1 - 2\nu_M} B_4 = bD_1 + \frac{3D_3}{b^4} + \frac{5 - 4\nu}{1 - 2\nu} D_4, \]
\[ bB_1 - \frac{7 - 4\nu_M}{1 - 2\nu_M} b^3 B_2 - \frac{2B_3}{b^4} + \frac{2D_3}{b^4} = bD_1 - \frac{2D_3}{b^4} + \frac{2D_4}{b^4}, \]
\[ \lambda_M \left( 21b^2 B_2 - \frac{6}{b^3} B_4 \right) + 2\mu_M \left( B_1 - \frac{18\nu_M}{1 - 2\nu_M} b^2 B_2 - \frac{12}{b^3} B_3 - \frac{5 - 4\nu_M}{1 - 2\nu_M} B_4 \right) \]
\[ = -6\lambda_1 D_4 + 2\mu \left( D_1 - \frac{12}{b^5} D_3 - \frac{5 - 4\nu}{1 - 2\nu} D_4 \right), \]
\[ \mu_1 \left( B_1 - \frac{7 + 2\nu_M}{1 - 2\nu_M} b^2 B_2 + \frac{8}{b^3} B_3 + \frac{2\nu_1 + \nu}{1 - 2\nu} B_4 \right) = \mu \left( D_1 + \frac{8}{b^5} D_3 + \frac{2\nu_1 + \nu}{1 - 2\nu} D_4 \right), \]

where only two of the effective properties \( \lambda, \mu, \nu \) are independent.

Next, we use Eshelby’s formula 4.19 with specified displacements on the outer boundary to evaluate the strain energy:

\[ U_{EQ} = U_0 + \frac{1}{2} \int_S \left( \sigma_i u_i^0 - \sigma_i^0 u_i \right) \text{d}s. \]
Here, the left-hand side of Eshelby’s formula is already adjusted to $U_{\text{EQ}}$, $S$ is the surface of the sphere $r = b$ and $U_0$ is the strain energy stored in the body completely consisting of the homogeneous material outside the composite sphere. Thus, the $U_0$ is the strain energy stored in the equivalent homogeneous media, i.e. it is the same as $U_{\text{EQ}}$:

$$U_{\text{EQ}} = U_0.$$  \hfill (4.110)

Therefore, the criterion for the determination of the effective properties is reduced to

$$0 = \frac{1}{2} \int_S (\sigma_i u_i^0 - \sigma_i^0 u_i) ds.$$  \hfill (4.111)

Following the same procedure to calculate the integral $4.111$ as in subsection 4.1.1, the equation $4.111$ reduces to

$$D_4 = 0.$$  \hfill (4.112)

So it is necessary to express formulae $4.101$ - $4.108$ for $D_4$ and set this expression to zero. This gives an equation for the effective shear modulus $\mu$. It can be shown that during these steps other effective properties $\lambda$ and $\nu$ are eliminated. Finally, the solution for $\mu$ is determined from the quadratic equation

$$A \left( \frac{\mu}{\mu_M} \right)^2 + 2B \frac{\mu}{\mu_M} + C = 0.$$  \hfill (4.113)

Here

$$A = 8 \left( \frac{\mu_1}{\mu_M} - 1 \right) (4 - 5\nu_M) \eta_1 c^{10/3} - 2 \left( 63 \left( \frac{\mu_1}{\mu_M} - 1 \right) \eta_2 + 2\eta_1 \eta_3 \right) c^{7/3} + 252 \left( \frac{\mu_1}{\mu_M} - 1 \right) \eta_2 c^{5/3} - 50 \left( \frac{\mu_1}{\mu_M} - 1 \right) (7 - 12\nu_M + 8\nu_M^2) \eta_2 c + 4(7 - 10\nu_M) \eta_2 \eta_3,$$  \hfill (4.114)

$$B = -2 \left( \frac{\mu_1}{\mu_M} - 1 \right) (1 - 5\nu_M) \eta_1 c^{10/3} + 2 \left( 63 \left( \frac{\mu_1}{\mu_M} - 1 \right) \eta_2 + 2\eta_1 \eta_3 \right) c^{7/3} - 252 \left( \frac{\mu_1}{\mu_M} - 1 \right) \eta_2 c^{5/3} + 75 \left( \frac{\mu_1}{\mu_M} - 1 \right) (3 - \nu_M) \eta_2 \eta_3 c + \frac{3}{2} (15\nu_M - 7) \eta_2 \eta_3,$$  \hfill (4.115)

$$C = 4 \left( \frac{\mu_1}{\mu_M} - 1 \right) (5\nu_M - 7) \eta_1 c^{10/3} - 2 \left( 63 \left( \frac{\mu_1}{\mu_M} - 1 \right) \eta_2 + 2\eta_1 \eta_3 \right) c^{7/3} - 252 \left( \frac{\mu_1}{\mu_M} - 1 \right) \eta_2 c^{5/3} + 25 \left( \frac{\mu_1}{\mu_M} - 1 \right) (\nu_M^2 - 7) \eta_2 c - (7 + 5\nu_M) \eta_2 \eta_3.$$  \hfill (4.116)

with

$$\eta_1 = (49 - 50\nu_M) \left( \frac{\mu_1}{\mu_M} - 1 \right) + 35 \frac{\mu_1}{\mu_M} (\nu_1 - 2\nu_M) + 35(2\nu_1 - \nu_M),$$

$$\eta_2 = 5\nu_1 \left( \frac{\mu_1}{\mu_M} - 8 \right) + 7 \left( \frac{\mu_1}{\mu_M} + 4 \right),$$

$$\eta_3 = \frac{\mu_1}{\mu_M} (8 - 10\nu_M) + (7 - 5\nu_M).$$
As in the previous subsections, \( c = \frac{(a/b)^3}{3} \) is the volume fraction of the inclusions.

In case of dilute spheres distribution, the solution of Eq. 4.113 for \( \mu \) can be reduced to the solution obtained in subsection “Dilute suspension” 4.1.1 as formula 4.52.

N.B. It is impossible to use simple formula 4.12 for this problem to calculate the effective shear modulus because a composite has two materials inclusion in the matrix which will not experience a uniform strain state.

There is an internal consistency in the derivations conducted in the current subsection: \( \mu \) was determined in the form uncoupled from \( k \) similar to the solution for \( k \) (uncoupled from \( \mu \)) for the composite spheres model, cf. subsection 4.1.4. This finding can provide indirect support to the earlier assumption that the present solution may be an exact one for the effective shear modulus of the composite spheres model. At least, we can claim that it is the solution for the three-phase model.

The result of the comparison between the experimental data for a suspension of spherical particle according to [2] and the result of the three-spheres model are presented in Fig. 4.7.

\[
\frac{E}{E_M} = \frac{9k\mu}{3k + \mu} \tag{4.117}
\]

is obtained by combination of 4.113 and 4.93 and plotted as “2” in Fig. 4.7.

Additionally, the prediction by rule of mixture \( E = E_Ic + E_M(1 - c) \) is plotted as ”1”.

![Figure 4.7: Effective modulus, glass spheres in polyester matrix.](image-url)
4.1.6. The self-consistent scheme for polycrystalline materials

4.1.7. A concentrated suspension model

Consider a concentrated (volume concentration close to the maximum) suspension of the identical spheres in a continuous matrix phase. This is a difficult problem, an exact solution of which cannot be found. We will apply the asymptotic methods and several approximations. Nevertheless, the solution remains completely rational and it does not become an empirical approach. The first idealization is the consideration of perfectly rigid spherical particles of single size in an incompressible matrix. We want to calculate the effective shear properties of the composite. Note that there is a similarity to the viscous fluid problem (when a liquid is considered instead of the elastic matrix). Another restriction is that we consider only cubical packing arrangement for the spherical particles. The reason for this is that the maximum volume fraction is equal to \( \pi/6 \) for cubical packing which corresponds to the experimental observations for loosely packed systems where \( c_{\text{max}} \) is slightly over one-half.

Cubical packing of single size spheres is shown in Fig. 4.8.

![Figure 4.8.: Cubical packing of concentrated suspension model.](image)

The spherical particles will be almost touching each other when the volume fraction is close to its maximum value \( c_{\text{max}} \). In such case the highest state of deformation occurs in the regions where the spheres are nearly in contact. From Fig. 4.8

\[
\Delta = l - 2a \quad \text{or} \quad \frac{\Delta}{a} = \frac{l}{a} - 2, \quad (4.118)
\]

where \( l \) is the dimension of the unit cell.

We can express the unit cell parameter \( l/a \) in terms of the volume fraction of the spheres as

\[
\frac{l}{a} = \frac{1}{2} \left( \frac{c}{c_{\text{max}}} \right)^{1/3}, \quad (4.119)
\]
As it was mentioned earlier, \( c_{\text{cubical packing}} = \pi/6 \).

Pure shear is specified by

\[
\epsilon_{xx} = \epsilon, \quad \epsilon_{yy} = 0, \quad \epsilon_{zz} = -\epsilon. \tag{4.120}
\]

Please, recall from subsection 4.1.1 that such state is imposed by a relative displacement of value \( u \) in one direction and \(-u\) in the orthogonal direction. The average strain is then equal to \( \epsilon = u/l \) as well as the maximum shear strain.

As in the previous subsections, we equate the strain energy stored in the composite cubical unit and in the equivalent homogeneous cubical unit. Strain energy of the homogeneous cubical unit is

\[
U = 2\mu l^3 \left( \frac{u}{T} \right)^2, \tag{4.121}
\]

where \( \mu \) is the effective shear modulus that we are trying to define.

It is difficult to derive an exact analytical solution for the energy in the composite structure (Fig. 4.8).

So we make another approximation to replace the problem in Fig. 4.8 (curved boundaries) with a simpler one. We have to solve this simpler problem of lubrication approximation (in terms of fluid behavior) shown in Fig. 4.9. To do this, let us start with a basic parallel boundary problem.

![Close packing approximation](image)

Figure 4.9.: Close packing approximation.

![Parallel boundary problem](image)

Figure 4.10.: Parallel boundary problem.

For a parallel boundary problem written in the cylindrical CS we can write

\[
u_z = \pm \frac{u}{2}, \quad u_r = 0, \quad \text{at } z = \pm \frac{h}{2}. \tag{4.122}
\]
Here, \( u \) is the relative displacement of the boundaries. Since we consider incompressible material,

\[
\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi} = 0,
\]

(4.123)

where

\[
\epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \quad \epsilon_{rz} = \frac{1}{2} \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right).
\]

(4.124)

The equilibrium equations are

\[
-\frac{\partial p}{\partial r} + \mu_M \nabla^2 u_r = 0,
\]

(4.125)

\[
-\frac{\partial p}{\partial z} + \mu_M \nabla^2 u_z = 0.
\]

(4.126)

Here \( p \) is the hydrostatic pressure supported by the incompressible material, \( \mu_M \) is the shear modulus of the material.

In such problems \( u_z \ll u_r \) and \( \frac{\partial u_r}{\partial r} \ll \frac{\partial u_r}{\partial z} \), so the equilibrium equations simplify to

\[
\mu_M \frac{\partial^2 u_r}{\partial z^2} = \frac{\partial p}{\partial r},
\]

(4.127)

\[
\frac{\partial p}{\partial z} = 0.
\]

(4.128)

The solution of the problem 4.127, 4.128, 4.123 taking into account 4.122 is

\[
u_r = \frac{3}{4} \frac{u}{h} \left( \frac{4z^2}{h^2} - 1 \right),
\]

(4.129)

\[
u_z = -\frac{3}{2} \frac{u}{h} \left( \frac{4z^3}{3h^2} - z \right).
\]

(4.130)

Integral of the local strain energy \( w \) over the thickness is

\[
\int_0^{h/2} \mu_M \epsilon_{ij} \epsilon_{ij} dz = \frac{9\mu_M}{10} \left( \frac{u}{h} \right)^2 h + \frac{3\mu_M}{4} \left( \frac{u}{h} \right)^2 \frac{r^2}{h}.
\]

(4.131)

It is assumed that 4.131 is the solution of the problem in Fig. 4.9 for each gap \( h \) along \( r \), i.e. \( h = h(r) \).

The total energy in each of four of the "half"-gaps (we are considering a 3D cubical packing) is

\[
U = 8\pi \mu_M \int_0^a \left( \frac{9}{10} \left( \frac{u}{h} \right)^2 h + \frac{3}{4} \left( \frac{u}{h} \right)^2 \frac{r^2}{h} \right) r dr.
\]

(4.132)

where

\[
\frac{h}{2} = \left( \frac{\Delta}{2} + a \right) - a \cos \theta,
\]

(4.133)

\[
r = a \sin \theta.
\]

(4.134)
When the gap $\Delta$ between spheres $\to 0$, we have to keep only the second term in the expression \(4.132\) for the energy (please analyze the behavior of both terms in the expression for $U$ when $h \to 0$), so

$$U = \frac{3\pi}{4} \mu_M u^2 a \int_0^{\pi/2} \frac{\sin^3 \theta \cos \theta d\theta}{(\Delta/2a + 1 - \cos \theta)^3}.$$  \hspace{0.5cm} (4.135)

Integrating the last equation and retaining the term with singular behavior when $\Delta \to 0$ leads to

$$U = \frac{3\pi}{2} \mu_M u^2 a \frac{a}{\Delta}.$$  \hspace{0.5cm} (4.136)

As usually, we are equating the energy in the suspension to the energy in the equivalent homogeneous media (formula \(4.121\)), so

$$\frac{\mu}{\mu_M} = \frac{3\pi a a}{4 \Delta I},$$  \hspace{0.5cm} (4.137)

and the subsequent substitution of \(4.118\) and \(4.119\) finally leads to

$$\frac{\mu}{\mu_M} = \frac{3\pi}{16} \frac{1}{1 - \left(\frac{c}{c_{\text{max}}}\right)^{1/3}}.$$  \hspace{0.5cm} (4.138)

Formula \(4.138\) for the effective shear modulus was derived under the assumption of $\Delta \to 0$, so it is valid for $c \to c_{\text{max}}$. Formula \(4.138\) specifies the order of singularity as $\frac{c}{c_{\text{max}}} \to 1$.

#### 4.1.8. General comments about the models for spherical particle composite systems and comparison of different methods

After analyzing all the above approaches, it can be concluded that only the composite spheres model and the self-consistent models have analytical characterization covering the entire volume fraction $c$ range.

The composite spheres model represents the behavior of many systems of practical interest. It gives good approximation even for the case of volume fraction $0.45 < c < 0.5$, cf. Fig. 4.7, i.e. in some cases it can characterize the behavior of concentrated single size spherical particles.

The composite spheres model can represent not only systems with exactly spherical inclusions, but the inclusions with shapes close to the spherical. This is due to the fact that the effective properties

**Dilute suspension (Eshelby solution / power series solution):**

bulk modulus (formula \(4.54\))

$$k = k_M + \frac{(k_1 - k_M)c}{1 + (k_1 - k_M)/(k_M + \frac{4}{3} \mu_M)}.$$  \hspace{0.5cm} (4.139a)

or

$$k = k_M + \frac{c(k_1 - k_M)(3k_M + 4\mu_M)}{3k_1 + 4\mu_M},$$  \hspace{0.5cm} (4.139b)
shear modulus (formula 4.52)

\[
\frac{\mu}{\mu_M} = 1 - \frac{15(1 - \nu_M) \left(1 - \frac{\mu}{\mu_M}\right) c}{7 - 5\nu_M + 2(4 - 5\nu_M) \frac{\mu}{\mu_M}}
\]

(4.140a)

or

\[
\mu = \mu_M + \frac{5c\mu_M (\mu_1 - \mu_M) (3k_M + 4\mu_M)}{3k_M (3\mu_M + 2\mu_1) + 4\mu_M (2\mu_M + 3\mu_1)}
\]

(4.140b)

(recall that \(\nu = \frac{3k - 2\mu}{6k + 2\mu}\))

**Mori-Tanaka:**

bulk modulus (the same as for composite spheres model - formula 4.93)

\[
\frac{k - k_M}{k_1 - k_M} = \frac{c}{1 + (1 - c) \frac{k_1 - k_M}{k_M + \frac{2}{5}\mu_M}}
\]

(4.141a)

or

\[
k = k_M + \frac{c(k_1 - k_M) (3k_M + 4\mu_M)}{3(1 - c)(k_1 - k_M) + 3k_M + 4\mu_M}
\]

(4.141b)

shear modulus

\[
\mu = \mu_M + \frac{5c\mu_M (\mu_1 - \mu_M) (3k_M + 4\mu_M)}{5\mu_M (3k_M + 4\mu_M) + 6(1 - c) (\mu_1 - \mu_M) (k_M + 2\mu_M)}
\]

(4.142)

**Self-consistent:**

bulk modulus

\[
k = k_M + \frac{c(k_1 - k_M) (3k + 4\mu)}{3k_1 + 4\mu}
\]

(4.143)

shear modulus

\[
\mu = \mu_M + \frac{5c\mu (\mu_1 - \mu_M) (3k + 4\mu)}{3k (3\mu + 2\mu_1) + 4\mu (2\mu + 3\mu_1)}
\]

(4.144)
4.2. Cylindrical and lamellar systems

In chapter 4.1 we successfully employed the concept of effective homogeneity to spherical inclusions. Now we apply the approaches similar to those presented in subsections of chapter 4.1 to composites with cylindrical and lamellar inclusions.

Usage of rigid spherical inclusions typically leads to the strength degradation (compared to the properties of the matrix material). This brings as well some stiffening effect, but it is not considerable. So, the actual reason to use the spherical inclusions is to reduce the cost of the material and to improve the dynamic properties.

Utilization of fiber reinforced materials, quite the contrary, improves both the stiffness and strength of the composite.

A dilute suspension of randomly oriented platelets is as well considered in the present chapter to illustrate the strong difference from the fiber systems.

Geometrically, we can consider the cylindrical and lamellar inclusions as limiting cases of prolate and oblate ellipsoids (or considering a very high $\frac{\text{maximum}}{\text{minimum}}$ ratio of the ellipsoid dimensions). Another approach is to consider an infinite extent, i.e. the fibers are uniform circular cylinders with an infinite length.

4.2.1. Modulus properties of transversely isotropic media

Consider a case when all the fibers are aligned, see Fig. 4.11. We can treat the composite as an effectively homogeneous if the fibers are randomly placed in space. In such case there is a symmetry in properties in the plane which is normal to the fibers direction, so we have a case of transversely isotropic material, cf. subsection xxx. It is necessary to determine effective properties of the homogeneous transversely isotropic material if the properties of the fibers and the matrix are specified.

When choosing axis 1 as a symmetry axis, the stress-strain relations are written in the following
As it was specified in subsection xxx, there are five independent constants $C_{11}, C_{12}, C_{22}, C_{23}, C_{66}$ corresponding to five independent effective properties. For convenience, the engineering properties are used instead of $C_{11}, C_{12}, C_{22}, C_{23}, C_{66}$ which are not the best choice when determining the material characteristics from experiments.

In case of uniaxial stress $\sigma_{11} \neq 0$ while $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$. From formulas 4.145 follows that

$$\sigma_{11} = \frac{C_{11}}{C_{22} + C_{23}} \epsilon_{11},$$

(4.148)

where the uniaxial modulus $E_{11}$ can be directly measured. The corresponding Poisson’s ratios $\nu_{12} = \nu_{13}$ are defined as

$$\nu_{12} = -\frac{\epsilon_{22}}{\epsilon_{11}},$$

(4.149)

$$\nu_{13} = -\frac{\epsilon_{33}}{\epsilon_{11}}.$$  

(4.150)

In the notation of the Poisson’s ratio the second subindex stands for the response direction and the first one - for the direction of the imposed loading. From formulas 4.145 follows that

$$\nu_{12} = \nu_{13} = \frac{C_{12}}{C_{22} + C_{23}},$$

(4.151)

Subtraction of Eqs. 4.145$_2$ and 4.145$_3$ leads to $\epsilon_{22} = \epsilon_{33}$.

Then from Eq. 4.145$_2$ (or 4.145$_3$):

$$C_{12}\epsilon_{11} + (C_{22} + C_{23})\epsilon_{22} = 0 \Rightarrow \epsilon_{22} = -\frac{C_{12}}{C_{22} + C_{23}} \epsilon_{11},$$

(4.146)

and finally

$$\sigma_{11} = C_{11}\epsilon_{11} - \frac{2C_{12}^2}{C_{22} + C_{23}} \epsilon_{11}.$$  

(4.147)
In case of special loading defined by $\epsilon_{22} = \epsilon_{33} \neq 0$, $\epsilon_{11} = 0$, the stresses are $\sigma_{22} = \sigma_{33} = \sigma$. From formulas 4.145 follows that

$$\sigma = (C_{22} + C_{23}) \epsilon,$$

where the plane strain bulk modulus $K_{23}$ can be directly measured.

The shear moduli are measured directly:

$$\mu_{12} = \mu_{31} = C_{66}, \quad \mu_{23} = \frac{1}{2} (C_{22} - C_{23}).$$

(4.153)

The expressions for the engineering moduli can be inverted:

$$C_{11} = E_{11} + 4\nu_{12}^2 K_{23},$$
$$C_{12} = K_{23} \nu_{12},$$
$$C_{22} = \mu_{23} + K_{23},$$
$$C_{23} = -\mu_{23} + K_{23},$$
$$C_{66} = \mu_{12}.$$

(4.154)

From the experiments we can measure not only $E_{11}, K_{23}, \nu_{12}(=\nu_{13}), \mu_{12}(=\mu_{31})$ and $\mu_{23}$.

Consider the case of uniaxial tension normal to the axis of symmetry, e.g. $\sigma_{22} \neq 0$ while $\sigma_{11} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$. From formulas 4.145 follows that

$$\sigma_{22} = \left( C_{22} + \frac{C_{12}^2 (C_{23} - C_{22}) + C_{23} (C_{12}^2 - C_{11}^2 C_{23})}{C_{11} C_{22} - C_{12}^2} \right) \epsilon_{22},$$

where the uniaxial modulus $E_{22}$ can be directly measured.

The corresponding Poisson’s ratios $\nu_{21}$ and $\nu_{23}$ are defined as

$$\nu_{21} = \frac{-\epsilon_{11}}{\epsilon_{22}},$$
$$\nu_{23} = \frac{-\epsilon_{33}}{\epsilon_{22}}.$$

(4.156)

(4.157)

From formulas 4.145 follows that

$$\nu_{21} = \frac{C_{12} (C_{22} - C_{23})}{C_{11} C_{22} - C_{12}^2},$$
$$\nu_{23} = \frac{C_{11} C_{23} - C_{12}^2}{C_{11} C_{22} - C_{12}^2}.$$

(4.158)

From the symmetry in the plane of isotropy follows

$$\nu_{31} = \nu_{21}, \quad \nu_{32} = \nu_{23}.$$

(4.159)

Poisson’s ratios $\nu_{21} \neq \nu_{21}$ and the following relation is easy to verify:

$$\frac{\nu_{12}}{E_{11}} = \frac{\nu_{21}}{E_{22}}.$$

(4.160)
Other important relations between the material properties:

\[ E_{22} = \frac{4\mu_{23}K_{23}}{K_{23} + \mu_{23} + 4\nu_{12}^2\mu_{23}K_{23}/E_{11}}, \]  
(4.161)

\[ \nu_{23} = \frac{K_{23} - \mu_{23} - 4\nu_{12}^2\mu_{23}K_{23}/E_{11}}{K_{23} + \mu_{23} + 4\nu_{12}^2\mu_{23}K_{23}/E_{11}}, \]  
(4.162)

\[ \nu_{21} = \frac{4\nu_{12}^2\mu_{23}K_{23}}{E_{11}(K_{23} + \mu_{23}) + 4\nu_{12}^2\mu_{23}K_{23}}, \]  
(4.163)

\[ \nu_{12}^2 = \left( -\nu_{23} - \frac{1}{4} \frac{E_{22}}{K_{23}} + \frac{1}{4} \frac{E_{22}}{\mu_{23}} \right) \frac{E_{11}}{E_{22}}. \]  
(4.164)

Consider now the compliance properties of the transversely isotropic fiber composite. Rewrite 4.145 in matrix notation:

\[ \{\sigma_i\} = [C_{ij}]\{\epsilon_j\}, \]  
(4.165)

where \( \{\sigma_i\} \) and \( \{\epsilon_j\} \) are vectors consisting of six elements, \([C_{ij}]\) is a 6 x 6 stiffness matrix.

Inverted form of the above strain-stress relations is

\[ \{\epsilon_j\} = [S_{ij}][\sigma_i], \]  
(4.166)

where \([S_{ij}]\) is a 6 x 6 compliance matrix defined as

\[ [S_{ij}] = \text{cofactor matrix of } [C_{ij}]. \]  
(4.167)

Both the stiffness and the compliance matrix are symmetric: \( C_{ij} = C_{ji} \) and \( S_{ij} = S_{ji} \).

Explicit expression for the compliance matrix \([S_{ij}]\) has the form

\[
[S_{ij}] = \begin{bmatrix}
\frac{1}{E_{11}} & -\frac{\nu_{21}}{E_{22}} & -\frac{\nu_{21}}{E_{22}} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & -\frac{\nu_{22}}{E_{22}} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_{11}} & \frac{1}{E_{22}} & \frac{1}{E_{22}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\nu_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\mu_{12}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_{12}}
\end{bmatrix}
\]  
(4.168)

For example,

\[ \epsilon_{11} = \frac{1}{E_{11}}\sigma_{11} - \frac{\nu_{21}}{E_{22}}\sigma_{22} - \frac{\nu_{21}}{E_{22}}\sigma_{33} \]  
(4.169)

since the strain \( \epsilon_{11} \) due to the action of \( \sigma_{11} \) is \( \epsilon_{11} = \frac{1}{E_{11}}\sigma_{11} \),

the stress \( \sigma_{22} \) will give the contribution \( \epsilon_{11} = -\nu_{21}\epsilon_{22} \) in the \( x_1 \)-direction (\( \epsilon_{11} = -\frac{\nu_{21}}{E_{22}}\sigma_{22} \)), the same for \( \sigma_{33} \).

Consider now the bounds on Poisson’s ratios.
From formula 4.162 follows

\[
\begin{align*}
&\text{if } \mu_{23} \to \infty \quad \text{then } \nu_{23} \to -1 \\
&\text{if } K_{23} \to \infty, E_{11} \to \infty \quad \text{then } \nu_{23} \to 1
\end{align*}
\] (4.170)

(4.171)

It can be shown from these results that

\[-1 \leq \nu_{23} \leq 1.\] (4.172)

In contrast to the isotropic material with an upper bound for the Poisson’s ratio equal to \(\frac{1}{2}\), the transversely isotropic material has an upper bound for the Poisson’s ratio equal to 1 and it can be easily achieved. Consider an incompressible material and suppose that it is very rigid (assume infinite rigidity) in the \(x_1\)-direction. Then pull the material along the \(x_2\)-direction. It must contract by the same amount along the \(x_3\)-direction as it is pulled along the \(x_1\)-direction since the material is incompressible, thus \(\nu_{23} = 1\).

Try to analyze the behavior of the material with a negative Poisson’s ratio.

Next, consider bounds on \(\nu_{12}\) and \(\nu_{21}\). We can rewrite formula 4.161 as

\[
\nu_{12}^2 = \frac{E_{11}}{E_{22}} - \frac{E_{11}}{4} \left( \frac{1}{K_{23}} + \frac{1}{\mu_{23}} \right).
\] (4.173)

All properties on the right-hand side of the equation are nonnegative. The maximum value for \(\nu_{12}^2\) can be reached when \(K_{23} \to \infty\) and \(\mu_{23} \to \infty\), so

\[
|\nu_{12}| \leq \left( \frac{E_{11}}{E_{22}} \right)^{1/2}
\] (4.174)

or (using 4.160)

\[
|\nu_{21}| \leq \left( \frac{E_{22}}{E_{11}} \right)^{1/2}.
\] (4.175)

For the most fiber-reinforced composites \(E_{22} \ll E_{11}\) so \(\nu_{12}\) can be even larger than 1, whereas \(\nu_{21} \ll 1\).

4.2.2. Composite cylinders model

As it was shown in subsection 4.2.1, it is necessary to determine five independent effective properties for the transversely isotropic material. We will try to find an analytical representation for the effective properties depending on the characteristics of the fibers, matrix and volume concentration of the fibers.

A geometrical model to use is the composite cylinders model (see Fig. 4.12) which is similar to the composite spheres model (cf. subsection 4.1.4). The CCM is actually a 2D analogue to a 3D CSM which was proposed by Hashin and Rosin in 1964.

Fibers are taken to be infinitely long circular cylinders surrounded with the continuous matrix phase. As in subsection 4.1.4, the radius of the fiber is \(a\) (\(a\) varies for each individual fiber) while the radius of the composite cylinder is \(b\). The relation \(a/b\) is constant for every composite cylinder. The sizes of the composite cylinders must vary to fill the whole composite structure.

The composite cylinders model is quite effective allowing to determine four out of five effective moduli for the RVE.
Let us determine the effective uniaxial modulus $E_{11}$.

We take $\epsilon_{11} = \epsilon$ while $\sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$.

Here coordinate 1 is in the direction of fibers.

Now we switch to a cylindrical CS. The following displacement field is assumed:

$$u_{rf} = A_f r,$$
$$u_{rM} = A_M r + \frac{B_M}{r},$$
$$u_z = \epsilon z.$$  \hspace{2cm} (4.176, 4.177, 4.178)

Here the subscript “f” stands for ”fiber”.

The solution 4.176-4.178 satisfies the equations of equilibrium.

The constants $A_f$, $A_M$ and $B_M$ have to be found from two continuity conditions at the interface and the boundary condition:

$$u_{rf} = u_{rM}, \sigma_{rf} = \sigma_{rM},$$
$$\sigma_{rM} = 0.$$  \hspace{2cm} (4.179, 4.180)
The effective uniaxial modulus $E_{11}$ is defined from the relation

$$\langle \sigma_{zz} \rangle = E_{11} \epsilon \quad (4.181)$$

by the following formula

$$E_{11} = \frac{1}{\pi b^2} \int_A \sigma_z(r) da. \quad (4.182)$$

"A" is the cross-sectional area (in plane 2-3) of the composite cylinder.

After completion of all the calculation in the previous formula the final result for the effective uniaxial modulus $E_{11}$ for a single composite cylinder has the following form:

$$E_{11} = cE_f + (1-c)E_M + \frac{4c(1-c)(\nu_f - \nu_M)^2\mu_M}{k_f \nu_f + \mu_M^2} + \frac{c\mu_M}{k_M + \mu_M^2} + 1. \quad (4.183)$$

Proof that the result 4.183 for a single composite cylinder corresponds to the whole RVE exactly follows to that described in section 4.1.4 for the bulk modulus. We use the theorem of minimum potential energy to show that the expression 4.183 is a lower bound for the effective uniaxial modulus $E_{11}$ for the whole RVE. Conversely, the upper bound for the effective uniaxial modulus $E_{11}$ for the whole RVE can be found imposing stresses as boundary conditions and applying the theorem of minimum complementary energy. It is found that upper and lower bounds are equal, so the solution 4.183 is the exact effective uniaxial modulus $E_{11}$ for the RVE.

It is worth to mention that formula 4.183 is very good approximated by the first two terms which represent the rule of mixtures, namely $E_{11} = cE_f + (1-c)E_M$.

**Poisson’s ratio $\nu_{12}$**

The loading scenario used to determine the effective uniaxial modulus can be used to find the effective Poisson’s ratio $\nu_{12}$.\(^3\)

The effective Poisson’s ratio $\nu_{12}$ is calculated as

$$\nu_{12} = \frac{-u_r|_{r=b}}{eb}, \quad (4.184)$$

where $-u_r(b)/b$ is the lateral strain and $\epsilon$ is the imposed axial strain.

After trivial transformations the effective Poisson’s ratio is found as

$$\nu_{12} = c\nu_f + (1-c)\nu_M + \frac{c(1-c)(\nu_f - \nu_M)}{k_f + \mu_f^2} \left( \frac{\mu_M}{k_M + \mu_M^2} - \frac{\mu_M}{k_f + \mu_f^2} \right) + \frac{c\mu_M}{k_M + \mu_M^2} + 1. \quad (4.185)$$

Formula 4.185 is as well good approximated by the rule of mixtures $\nu_{12} = c\nu_f + (1-c)\nu_M$.

**Plane strain bulk modulus $K_{23}$**

The plane strain bulk modulus $K_{23}$ is determined in a similar manner by imposing the plane state loading as specified in subsection 4.2.1

$$K_{23} = k_M + \frac{\mu_M}{3} + \frac{c}{k_f - k_M + (\mu_f - \mu_M)/3} + \frac{(1-c)}{k_M + \mu_M^2} \quad (4.186)$$

\(^3\)It is necessary to remind that $\nu_{12} \neq \nu_{21}$, "1" is a fiber direction, the loading is applied in "1"-direction, the deformations are measured in "2"-direction.
Shear moduli in the fiber direction $\mu_{12} = \mu_{31}$

The shear modulus in the fiber direction $\mu_{12}$ is measured directly:

$$\frac{\mu_{12}}{\mu_M} = \frac{\mu_f (1 + c) + \mu_M (1 - c)}{\mu_f (1 - c) + \mu_M (1 + c)}. \quad (4.187)$$

The problems for the effective plane strain bulk modulus $K_{23}$ and the shear moduli in the fiber direction $\mu_{12} = \mu_{31}$ are simple since the solution $4.176 - 4.177$ contains only one variable $r$ as for the uniaxial modulus $E_{11}$ and the Poisson’s ratio $\nu_{12}$.

There are difficulties with the determination of the transverse shear modulus $\mu_{23}$ in the framework of the composite cylinders model which are similar to the problem for determination of the effective shear modulus $\mu$ in the framework of the composite spheres model, cf. subsection 4.1.4.

To determine the lower bound for $\mu_{23}$ it is necessary to apply the displacement boundary conditions and for the upper bound - the stresses. The determined lower and upper bounds do not coincide, except for the cases of the dilute distribution of fibers and very high volume fraction.

4.2.3. A model for the transverse shear of a fiber system (a three-phase model, generalized self-consistent approach)

Quite similar to the case of spherical inclusions and following the generalized self-consistent approach we consider a three-phase model to determine the effective transverse shear modulus $\mu_{23}$ of a fiber system [3]. The discussion about the applicability of this approach to determine $\mu_{23}$ for the composite cylinders model is exactly the same as in subsection 4.1.5.

We will replace all the single composite cylinders (except for one) by the equivalent homogeneous media, the properties of which are yet unknown and will be determined as the solution of the problem (Fig. 4.13).

In plane 2-3 simple shear deformation is applied. The state of deformation is taken as

$$U_r = \frac{b}{4\mu_f} \left( \frac{r}{b} A_1 + (\eta_f - 3) \frac{r^3}{b^3} A_2 \right) \cos 2\theta, \quad (4.188)$$

$$U_\theta = \frac{b}{4\mu_f} \left( -\frac{r}{b} A_1 + (\eta_f + 3) \frac{r^3}{b^3} A_2 \right) \sin 2\theta, \quad (4.189)$$

$$U_{rM} = \frac{b}{4\mu_{23}} \left( \frac{r}{b} B_1 + (\eta_M - 3) \frac{r^3}{b^3} B_2 + \frac{b^3}{r^3} B_3 + (\eta_M + 1) \frac{b}{r} B_4 \right) \cos 2\theta, \quad (4.190)$$

$$U_{\theta M} = \frac{b}{4\mu_{23}} \left( -\frac{r}{b} B_1 + (\eta_M + 3) \frac{r^3}{b^3} B_2 + \frac{b^3}{r^3} B_3 - (\eta_M - 1) \frac{b}{r} B_4 \right) \sin 2\theta, \quad (4.191)$$

$$U_{rEQ} = \frac{b}{4\mu_{23}} \left( \frac{2r}{b} + \frac{b^3}{r^3} D_3 + (\eta + 1) \frac{b}{r} D_4 \right) \cos 2\theta, \quad (4.192)$$

$$U_{\theta EQ} = \frac{b}{4\mu_{23}} \left( -\frac{2r}{b} + \frac{b^3}{r^3} D_3 - (\eta - 1) \frac{b}{r} D_4 \right) \sin 2\theta. \quad (4.193)$$
Here $\eta = 3 - 4\nu$, $\eta_M = 3 - 4\nu_M$, $\eta_f = 3 - 4\nu_f$. 
$\mu$ and $\nu$ are the unknown effective properties of the equivalent homogeneous media. 
When $r \to \infty$ the formulae 4.188 and 4.189 represent an imposed state of simple shear deformation in polar coordinates.

4.2.4. Finite length fiber effects

This theory has been developed to relate the effective elastic modulus to the aspect ratio and the axial geometry of the fibers. Using slender body theory, the problem of a deformable slender inclusion embedded in an infinite elastic inclusion medium which is uniformly strained at infinity can be solved, if

i) the equilibrium equations are solved for the displacements and stresses within the inclusions and

ii) the normal tractions $\sigma_{ij} n_j$ and the displacements $u_i$ are required to be continuous across the inclusion-matrix interface.

At dilute concentrations, the longitudinal Young’s modulus, longitudinal poisson’s ratio and the plane strain bulk modulus can be determined from the response of a single slender inclusion to an applied radially-symmetric tri-axial field.

Eshelby’s solution for the response a single ellipsoidal inclusion to an arbitrary homogeneous applied strain by first considering the misfitting inclusion of same material as the matrix has been used to calculate the final strain $e_{ij}^c$ as

$$ e_{ij}^c = S_{ijlm} e_{lm}^* $$

where $S_{ijlm}$ is the Eshelby tensor which depends on the elastic constants of the matrix and a,b,c
the axes of the ellipsoid and $e_{lm}^*$ is the eigen strain required to return the inclusion to the original shape. If an applied strain is superimposed on the system, the strain in the inclusion is given by

$$e_{ij} = e_{ij}^A + e_{ij}^c$$

and the stress is given as

$$\sigma_{ij} = \lambda(e_{kk}^A + e_{kk}^c)\delta_{ij} + 2\mu(e_{ij}^A + e_{ij}^c - e_{ij}^*)$$

If the misfitting inclusion is now replaced by one having different elastic properties, the $e_{ij}^*$ can be determined by condition

$$\lambda_I(e_{kk}^A + e_{kk}^c)\delta_{ij} + 2\mu_I(e_{ij}^A + e_{ij}^c - e_{ij}^*) = \lambda_M(e_{kk}^A + e_{kk}^c - e_{kk}^*) + 2\mu_M(e_{ij}^A + e_{ij}^c - e_{ij}^*)$$

By substituting the equation (1) in equation (3), we get

$$e_{ij}^* = A_{ijkl}e_{kl}^A$$

where $A_{ijkl}$ depends on the $S_{ijkl}$ and the elastic constants of both the phases. To solve for the effective properties of the heterogeneous media, we need to perform the averaging theorems. From which we obtain the following equation.

$$C_{ijkl}e_{kl}^A = \lambda_M\delta_{ij}e_{kk}^A + 2\mu_M e_{ij}^A + \frac{1}{V} \sum_{n=1}^{N} \int_{V_n} (\sigma_{ij} - \lambda_M e_{kk}\delta_{ij} - 2\mu_M e_{ij}) dv.$$  

by substituting the equation (2) in equation (6), we get

$$C_{ijkl}e_{kl}^A = \lambda_M\delta_{ij}e_{kk}^A + 2\mu_M e_{ij}^A - c(\lambda_M e_{ij}^* \delta_{ij} + 2\mu_M e_{ij}^*)$$

By considering the case of shear loading $e_{ji}^A = e_{ij}^A$, we get

$$e_{23}^* = \frac{(\mu_M - \mu)}{\mu_M + 2(\mu_M - \mu)S_{2323}} e_{12}^A$$

by substituting the above equation in equation (7)

$$\frac{\mu_{23}}{\mu_M} = 1 + \frac{c \left( \frac{\mu}{\mu_M} - 1 \right)}{1 + 2 \left( \frac{\mu}{\mu_M} - 1 \right) S_{2323}}$$

where $S_{2323} = \frac{3-4\nu_M}{2(1-\nu_M)}$. If the ratio $k$ of radius of inclusion to the length of the inclusion is very much less than one, $k = \frac{b}{a} \ll 1$ the Young’s modulus along the fiber length will be equal to

$$\frac{E_{11}}{E_M} = 1 + c \frac{\Delta \mu}{2(1-\nu_M)}(3\Delta \lambda + 2\Delta \mu) + E_M \frac{(1-2\nu_M)}{2(1-\nu_M)} (\Delta \lambda + \frac{3\lambda_M}{3\lambda_M + 2\mu_M})$$

$$\Delta \mu = \frac{\Delta \mu}{2(1-\nu_M)}(3\Delta \lambda + 2\Delta \mu) + E_M \frac{(1-2\nu_M)}{2(1-\nu_M)} (\Delta \lambda + \frac{3\lambda_M}{3\lambda_M + 2\mu_M})$$

$$+ \frac{5-4\nu_M}{4(1-\nu_M)} k^2 \left( \ln 2/k - \frac{5-4\nu_M}{4(1-\nu_M)} \Delta \lambda + \frac{3\lambda_M}{3\lambda_M + 2\mu_M} \right).$$
where $\Delta \lambda = \lambda_I - \lambda_M$ and $\Delta \mu = \mu_I - \mu_M$

$$\frac{E_{11}}{E_M} = 1 + c \frac{O(E_I^2) + O(E_I) + O(E_I)}{O(E_I^2) + O(E_I) + O(E_I)}$$

(4.204)

To examine the two asymptotes of the above equation, the following cases are considered:

Case 1:

$$\frac{E_L}{E_M} k^2 \ln k / 2 \ll 1$$

only the quadratic terms are considered from equation (11) and all other terms are neglected.

Case 2:

$$\frac{E_L}{E_M} k^2 \ln k / 2 \gg 1$$

only the linear terms from equation (11) are considered since all other terms are negligible.

Figure 4.14.: Finite length fiber effects according to [4].

4.2.5. A dilute suspension of randomly oriented platelets

Consider a randomly oriented platelets in infinite matrix phase. The physical meaning of dilute suspension is that the particles are small and so far away of each other that all interactions between them can be neglected. Considering the dilute suspension in a state of imposed simple shear deformation at larger distances from the inclusion, we get

$$\frac{\mu - \mu_M}{\mu_I - \mu_M} = c \frac{\langle \epsilon \rangle}{\langle \epsilon \rangle}.$$

(4.205)
where \( c \) is the volume fraction of the inclusions.
The bulk modulus \( k \) can be derived in a similar manner as
\[
\frac{k - k_M}{k_I - k_M} = c \left( \frac{\langle \epsilon_i^I \rangle}{\langle \epsilon \rangle} \right).
\] (4.206)

The total strain in the inclusion can be given as
\[
\epsilon_{ij}^I = \epsilon_{ij}^A + \epsilon_{ij}^c,
\] (4.207)
\[
\epsilon_{ij}^c = S_{ijkl} \epsilon_{kl}^*.
\] (4.208)

From the equivalent inclusion method, we get eigen strain equal to
\[
\epsilon_{ij}^* = A_{ijkl} \epsilon_{kl}^A.
\] (4.209)

By substituting the above equation in equation (4), the constrained strain in the matrix - inclusion will be equal to
\[
\epsilon_{ij}^c = S_{ijkl} A_{klmn} \epsilon_{mn}^A.
\] (4.210)

The above equation can be rewritten as
\[
\epsilon_{ij}^c = T_{ijmn} \epsilon_{mn}^A.
\] (4.211)

Under hydrostatic stress condition, the applied strain will be equal to
\[
\dot{\epsilon}_{ij} = \frac{\epsilon}{3} \delta_{ij}.
\] (4.212)

and the strain in the inclusion under hydrostatic condition
\[
\epsilon_{ii}^I = \frac{T_{iijj}}{3} \epsilon_{jj}.
\] (4.213)

Since the oblate ellipsoid has the same axes, \( b = c \) with \( a < b \), and coordinate direction 1 corresponds to the axis of symmetry of the ellipsoid only the limiting case behaviour \( a/b \to 0 \) is explored.
\[
\lim_{a/b \to 0} I_2 = 0
\] (4.214)

and from Eshelby, it follows that
\[
I_1 = 4\pi, I_2 = I_3 = 0.
\] (4.215)

By substituting the value of \( S_{ijkl} \) in the above equations we get the final result as
\[
\frac{k - k_M}{k_I - k_M} + \frac{c}{1 + (k_I - k_M)/(k_M + \frac{4}{3} \mu_I)}.
\] (4.216)

In the case effective shear modulus \( \mu \), the state of shear deformation can be specified by
\[
\epsilon_{ij}^0 = \epsilon_{ij}^0, \epsilon_{kk}^0 = 0
\] (4.217)

The shear strain in the inclusion can be given as
\[
(\epsilon_{ij})_{INCL} = T_{ijkl} \epsilon_{kl}^0 - \frac{T_{ijkl}}{3} \epsilon_{kl}^0 \delta_{ij}.
\] (4.218)
The volumetric average of \((e_{ij})_{INCL}\) over all possible orientations of ellipsoidal inclusion is designated by \(\langle e_{ij} \rangle\). This latter quantity must be linear function of the invariants of first degree of \(T_{ijkl}\), namely, \(T_{ijij}\) and \(T_{jjkk}\). To find the linear function of these invariants we begin by writing the specific forms for the equation.

\[
\frac{(e_{ij})_{0}}{\epsilon_{ij}} = C, \tag{4.219}
\]

where \(C\) is the constant.

\[
T_{1212}^{AVG} + \frac{C}{2}, \tag{4.220}
\]

where \(T_{ijkl}^{AVG}\) is the integration of \(T_{ijkl}\) over all possible orientations of the ellipsoid. Two more equations of this type follow from similar shear deformations in the other two planes. Next a shear deformation of the type \(\epsilon_{22}^{A} = -\epsilon_{11}^{A}\) is specified.

\[
T_{1111}^{AVG} + T_{2222}^{AVG} - T_{1122}^{AVG} - T_{2211}^{AVG} = 2C, \tag{4.221}
\]

\[
3T_{ijij}^{AVG} - T_{jjkk}^{AVG} = 15C. \tag{4.222}
\]

This is the desired result since \(T_{ijij}^{AVG} = T_{ijij}\) and \(T_{jjkk}^{AVG} = T_{jjkk}\) because of the invariant properties of the \(T_{ijkl}\). Therefore we can thus write

\[
\frac{(e_{ij})_{0}}{\epsilon_{ij}} = \frac{1}{15}(3T_{ijij} - T_{jjkk}). \tag{4.223}
\]

Proceeding as with the evaluation of \(k\), \(T_{ijij}\) and \(T_{jjkk}\) are found for the limiting case of the oblate ellipsoid, and when combined with equation (1) and equation (19), we get

\[
\frac{\mu - \mu_{M}}{\mu_{I} - \mu_{M}} = \frac{c}{1 + (\mu_{I} - \mu_{M})/(\mu_{M} + \mu_{d})}. \tag{4.224}
\]
5. Energetic bounds on effective properties

In the previous chapters we obtained several important results on the effective properties of the heterogeneous materials with spherical, cylindrical and lamellar inclusions. Now we try to get the upper and lower bounds for effective properties without specifying certain geometry of the inclusions. Please recall that we already determined the upper and lower bounds for some cases of two phase heterogeneous materials with inclusions of the particular geometric form (see Chapter 4).

5.1. Reuss lower and Voigt upper bounds

Consider the simple case of admissible stress (or strain) field which is uniform across all \( N \) phases of the composite material. The key in establishing the bounds is to find an appropriate admissible field. For the Voigt upper bound it is a kinematically admissible displacement field in the potential energy, while for the Reuss lower bound it is a statically admissible stress field in the complimentary energy.

Assume the uniform dilatation or simple shear strain across all phases which is imposed by linearly varying displacements. Then utilize minimum potential energy theorem (section 1.1). From energy equivalence used for definition of effective properties follows the Voigt upper bound:

\[
k \leq \sum_{r=1}^{N} c_r k_r, \quad \mu \leq \sum_{r=1}^{N} c_r \mu_r.
\]

(5.1)

As before, \( V_r \) and \( c_r \) are the volumes of individual phases and volume fractions, respectively.

Now do the same in case of uniform state of stress imposed to the entire composite and utilize minimum complementary energy theorem (section 1.1) to determine the Reuss lower bound:

\[
\frac{1}{k} \leq \sum_{r=1}^{N} \frac{c_r}{k_r}, \quad \frac{1}{\mu} \leq \sum_{r=1}^{N} \frac{c_r}{\mu_r}.
\]

(5.2)

Combining both relations gives

\[
\sum_{r=1}^{N} \frac{c_r}{k_r} \leq k \leq \sum_{r=1}^{N} c_r k_r, \quad \sum_{r=1}^{N} \frac{c_r}{\mu_r} \leq \mu \leq \sum_{r=1}^{N} c_r \mu_r.
\]

(5.3)

Unfortunately, there is a big discrepancy between the upper and lower bounds defined by 5.3 - 5.4. Please notice that we did not specify geometry of phases to derive these equations. If to choose the shape of the inclusions, it is possible to obtain more explicit expressions for the effective properties.
5.2. **Extension of classical variational principles: Hashin-Shtrikman bounds**

Tighter bounds (without specifying inclusions geometry) were found by Hashin and Shtrikman [5] by applying kinematically (statically) admissible fields which make explicit allowance for microstructure of the heterogeneous material. Walpole [6] generalized the results of [5]. It should be mentioned that works of R. Hill of that time were used in derivations.

We follow the work [6] of Walpole.

**The upper bound.**

The displacement boundary conditions are applied to the surface of a composite material. We will use tensor notation instead of index notation in this section.

As in section 2.1 (Eshelby’s formula) there is a homogeneous comparison material which is subjected to the same boundary conditions. The elastic moduli of the comparison material are $C_0$ whereas each phase $r$ of the composite has the elastic moduli $C_r$.

Let us designate any strain field (resulted from the displacement field which satisfies the conditions at the surface) in the comparison material as $\varepsilon$ and construct the following expression:

$$\sigma^* = C_0 \varepsilon + \tau.$$  \hspace{1cm} (5.5)

The stress $\sigma^*$ satisfies the equilibrium equation (is self-equilibrated) which will be confirmed later. Only the polarization stress

$$\tau = (C_r - C_0) \varepsilon$$  \hspace{1cm} (5.6)

gives the exact solution for the heterogeneous material since in such case

$$\sigma^* = C_r \varepsilon.$$  \hspace{1cm} (5.7)

We will use a piecewise constant form of the polarization stress:

$$\tau = (C_r - C_0) \bar{\varepsilon}_r.$$  \hspace{1cm} (5.8)

Here $\bar{\varepsilon}_r$ is the average of $\varepsilon$ over the volume $V_r$.

Consequently, Eq. (5.5) takes the form

$$\sigma^* = C_r \bar{\varepsilon}_r + C_0 (\varepsilon - \bar{\varepsilon}_r) .$$  \hspace{1cm} (5.9)

$\varepsilon'_r$ is the deviation of strain in the volume $V_r$ from the average strain in this volume.

According to section 1.1 the theorem of minimum potential energy defines

$$U \leq \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \varepsilon C_r \varepsilon dv,$$  \hspace{1cm} (5.10)

where $\varepsilon$ is defined in Eq. (5.5) as any admissible strain field and $U$ is the strain energy which corresponds to the exact solution.
Now we modify the last equation without changing its value by adding the second term to the right-hand side:

\[
U \leq \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \epsilon dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \sigma^*(\bar{\epsilon} - \epsilon).
\] (5.11)

\(\bar{\epsilon}\) is the volume average of \(\epsilon\).

\(\sigma^*\) satisfies the equilibrium equations \(\Rightarrow \sigma^*(\bar{\epsilon} - \epsilon)\) is the expression of the principle of virtual work \(\Rightarrow \sigma^*(\bar{\epsilon} - \epsilon) = 0\).

Substituting [5.9] in Eq. 5.11 leads to

\[
U \leq \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \epsilon dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} (\bar{\epsilon} - \epsilon) \left( C_r (\bar{\epsilon} - \epsilon_r') + C_0 \epsilon_r' \right) dv
\] (5.12)

or, transforming,

\[
\frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \epsilon dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} (\bar{\epsilon} - \epsilon) \left( C_r (\bar{\epsilon} - \epsilon_r') + C_0 \epsilon_r' \right) dv
\]

\[
= \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \epsilon dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} (\bar{\epsilon} - \epsilon) C_r \bar{\epsilon} dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} (\bar{\epsilon} - \epsilon) \left( -C_r \epsilon_r' + C_0 \epsilon_r' \right) dv
\]

\[
= \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \epsilon dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \bar{\epsilon} C_r \epsilon dv - \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \bar{\epsilon} dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \left( \bar{\epsilon} - \epsilon \right) \left( C_r - C_0 \right) \epsilon_r' dv
\]

\[
= \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \epsilon dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \bar{\epsilon} C_r \epsilon dv - \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r \bar{\epsilon} dv - \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \bar{\epsilon} C_r \epsilon_r' dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \bar{\epsilon} C_0 \epsilon_r' dv
\]

\[
= \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} (\bar{\epsilon} - \epsilon) C_r (\bar{\epsilon} - \epsilon) dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \bar{\epsilon} C_r \epsilon_r' dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} [\bar{\epsilon} C_0 + \bar{\epsilon} (C_r - C_0)] \epsilon_r' dv
\]

\[
+ \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r (C_r - C_0) \epsilon_r' dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon C_r (C_r - C_0) \epsilon_r' dv
\]

\[
= \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} (\bar{\epsilon} - \epsilon) C_r (\bar{\epsilon} - \epsilon) dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \bar{\epsilon} C_r \epsilon_r' dv + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} [\bar{\epsilon} C_0 + \bar{\epsilon} (C_r - C_0)] \epsilon_r' dv
\]

The first term is equal to zero since we integrate over the whole volume and \(\bar{\epsilon}\) is the volume average of \(\epsilon\). The third term is equal to zero since the averages can be extracted from the integral leaving \(\int_{V_r} \epsilon_r' dv = 0\) according to the definition of \(\epsilon_r'\) as the deviation of strain in the volume \(V_r\) from the average strain in this volume.
Finally, we get

\[
U \leq \frac{1}{2} \sum_{r=1}^{N} V_r \bar{\epsilon}_r \bar{\epsilon}_r + \frac{1}{2} \sum_{r=1}^{N} \int_{V_r} \epsilon'_r (C_r - C_0) \epsilon'_r dv. \tag{5.13}
\]

The integration in the first term was performed directly since it contained only averages.
In contrast to the second term, the first term is not symmetric, because it involves both $\bar{\epsilon}$ and $\bar{\epsilon}_r$.

Let us now define a new tensor
Please refer to the excerpt on localization tensors below.
For example, in Eshelby dilute approach $A_r = T_r$ at each inhomogeneity, cf. section 3.3.
Localization tensors

We can write a general expression for disordered heterogeneous materials connecting the local strains \( \epsilon(x) \) and stresses \( \sigma(x) \) with the macroscopic strains \( \bar{\epsilon} \) and stresses \( \bar{\sigma} \):

\[
\epsilon(x) = A(x)\bar{\epsilon}, \quad \sigma(x) = B(x)\bar{\sigma}.
\]

(5.14)

(5.15)

Tensors \( A(x) \) and \( B(x) \) are called strain and stress localization tensors, respectively.

In case of \( N \) distinct phases, the average strains \( \epsilon_r \) and stresses \( \sigma_r \) in each phase \( r \) are connected with the macroscopic strains \( \bar{\epsilon} \) and stresses \( \bar{\sigma} \) as

\[
\epsilon_r = A_r\bar{\epsilon}, \quad \sigma_r = B_r\bar{\sigma}.
\]

(5.16)

(5.17)

Here

\[
\epsilon_r = \frac{1}{V_r}\int_{V_r} \epsilon(x)dv, \quad \sigma_r = \frac{1}{V_r}\int_{V_r} \sigma(x)dv
\]

(5.18)

(please compare with the definition of the average strain and stress fields in RVE, Eqs. 4.1 - 4.2).

From the above equations follows that

\[
\frac{1}{V}\int_{V} A(x)dv = I, \quad \frac{1}{V}\int_{V} B(x)dv = I
\]

(5.19)

in case of disordered heterogeneous material or

\[
\sum_{r=1}^{N} c_r A_r = I, \quad \sum_{r=1}^{N} c_r B_r = I
\]

(5.20)

in case of \( N \) distinct phases.

N.B. In many literature sources the matrix is indexed as '0' and different types of inclusions have indexes ranging from '1' to 'N'. In such case the summation in the previous equation will start from zero as

\[
\sum_{r=0}^{N} c_r A_r = I, \quad \sum_{r=0}^{N} c_r B_r = I.
\]

(5.21)

Let us derive the relation between the stress and strain localization tensors. The constitutive law in the phase \( r \) with a stiffness tensor \( C_r \) can be written as

\[
\sigma_r = C_r\epsilon_r \quad \text{or} \quad B_r\bar{\sigma} = C_r A_r \bar{\epsilon}.
\]

(5.22)

Substitution of the constitutive law \( \bar{\sigma} = C\bar{\epsilon} \) for the homogenized composite media in the last equation finally leads to

\[
B_r\bar{\epsilon} = C_r A_r \bar{\epsilon}.
\]

(5.23)
5.2.1. Hashin-Shtrikman bounds on transversely isotropic effective moduli

Plane strain bulk modulus

\[ k_2 + \frac{c_1}{k_1 - k_2} + \frac{c_2}{k_2 + \mu_2} \leq k_{23} \leq k_1 + \frac{c_2}{k_2 - k_1} + \frac{c_1}{k_1 + \mu_1}. \]  

(5.24)

Shear modulus

\[ \mu_2 + \frac{c_1}{\mu_1 - \mu_2} + \frac{c_2(\mu_1 + 2\mu_2)}{2\mu_2(\mu_1 + 2\mu_2)} \leq \mu_{23} \leq \mu_1 + \frac{c_2}{\mu_2 - \mu_1} + \frac{c_1(\mu_1 + 2\mu_1)}{2\mu_1(\mu_1 + \mu_1)}. \]  

(5.25)

Uniaxial modulus

\[ \frac{c_1c_2}{\mu_1 - \mu_2} + \frac{c_2}{\mu_2} \leq \frac{E_{11} - c_1E_1 - c_2E_2}{4(\nu_1 - \nu_2)^2} \leq \frac{c_1c_2}{\mu_1 - \mu_2} + \frac{c_2}{\mu_2}. \]  

(5.27)

Poisson’s ratio

\[ \frac{c_1c_2}{\mu_1 - \mu_2} + \frac{1}{\mu_2} \leq \frac{\nu_{12} - c_1\nu_1 - c_2\nu_2}{(\nu_1 - \nu_2)(1 - \frac{1}{k_1})} \leq \frac{c_1c_2}{\mu_1 - \mu_2} + \frac{1}{\mu_2}. \]  

(5.28)
6. Isotropic properties of randomly oriented fibers

6.1. 3D case

A system with completely three-dimensional random orientation of fibers in an isotropic matrix phase.

\( \{x'_1, x'_2, x'_3\} \) - fixed coordinate system, \( x'_3 \) - direction of the applied loading,
\( \{x_1, x_2, x_3\} \) - random coordinate system for a fiber orientation along \( x_1 \) axis.

The transformation law for the strain tensor:

\[
\epsilon_{\alpha\beta} = l_{\alpha i} l_{\beta j} \epsilon'_{ij} \tag{6.1}
\]

with the rotation tensor

\[
l_{\alpha i} = \begin{bmatrix}
\sin \theta \cos \phi & -\cos \theta \cos \phi & \sin \phi \\
\sin \theta \sin \phi & -\cos \theta \sin \phi & -\cos \phi \\
\cos \theta & \sin \theta & 0
\end{bmatrix}. \tag{6.2}
\]

Suppose that only \( \epsilon'_{33} \neq 0 \).

Then substitution of Eq. 6.1 in Eq. 4.145 leads to

\[
\frac{\sigma_{11}}{\epsilon_{33}} = C_{11} l_{31}^2 + C_{12} l_{32}^2, \quad \frac{\sigma_{22}}{\epsilon_{33}} = C_{12} l_{31}^2 + C_{22} l_{32}^2, \quad \frac{\sigma_{33}}{\epsilon_{33}} = C_{12} l_{31}^2 + C_{23} l_{32}^2, \quad \frac{\sigma_{12}}{\epsilon_{33}} = 2C_{66} l_{31} l_{32}, \tag{6.3}
\]

Using the transformation law for the stress tensor

\[
\sigma'_{\alpha\beta} = l_{\alpha i} l_{\beta j} \sigma_{ij}, \tag{6.4}
\]

gives

\[
\frac{\sigma'_{33}}{\epsilon_{33}} = C_{11} l_{31}^4 + (2C_{12} + 4C_{66}) l_{31}^2 l_{32}^2 + C_{22} l_{32}^4, \tag{6.5}
\]
\[
\frac{\sigma'_{22}}{\epsilon_{33}} = C_{11} l_{31}^2 l_{21}^2 + C_{12} l_{32}^2 l_{21}^2 + l_{31}^2 l_{22}^2 + l_{31} l_{23}^2 + C_{22} l_{32}^2 l_{22}^2 + 4C_{66} l_{31} l_{32} l_{21} l_{22} + C_{23} l_{32} l_{23}^2. \tag{6.6}
\]

Average value:

\[
\frac{\sigma'_{ij}}{\epsilon_{33}} \bigg|_{\text{random}} = \frac{1}{2\pi} \int_0^\pi \int_0^\pi \frac{\sigma'_{ij}}{\epsilon_{33}} \sin \theta d\theta d\phi \tag{6.7}
\]

resulting in

\[
\frac{\sigma'_{33}}{\epsilon_{33}} \bigg|_{\text{random}} = \frac{1}{15} (3C_{11} + 4C_{12} + 8C_{22} + 8C_{66}), \tag{6.8}
\]
\[
\frac{\sigma'_{22}}{\epsilon_{33}} \bigg|_{\text{random}} = \frac{1}{15} (C_{11} + 8C_{12} + C_{22} - 4C_{66} + 5C_{23}). \tag{6.9}
\]
Since the right-hand sides of the above equations in case of effectively isotropic medium are equal to $k + \frac{k}{3}\mu$ and $k - \frac{2}{3}\mu$, respectively, the isotropic properties of fiber system are resolved as:

\[
k = \frac{1}{9}[C_{11} + 2(C_{22} + C_{23}) + 4C_{12}], \quad (6.10)
\]
\[
\mu = \frac{1}{30}[2C_{11} + 7C_{22} - 5C_{23} - 4C_{12} + 12C_{66}] \quad (6.11)
\]

or, alternatively, in engineering constants:

\[
k = \frac{1}{9}[E_{11} + 4(1 + \nu_{12})^2K_{23}], \quad (6.12)
\]
\[
\mu = \frac{1}{15}[2E_{11} + (1 - 2\nu_{12})^2K_{23} + 6(\mu_{12} + \mu_{23})]. \quad (6.13)
\]

$E$ and $\nu$ in engineering constants are defined as (cf. subsection [A] for relation between elastic constants):

\[
E = \frac{[E_{11} + 4(1 - \nu_{12})^2K_{23}][E_{11} + (1 - 2\nu_{12})^2K_{23} + 6(\mu_{12} + \mu_{23})]}{3[2E_{11} + (8\nu_{12}^2 + 12\nu_{12} + 7)K_{23} + 2(\mu_{12} + \mu_{23})]}, \quad (6.14)
\]
\[
\nu = \frac{E_{11} + 2(2\nu_{12}^2 + 8\nu_{12} + 3)K_{23} - 4(\mu_{12} + \mu_{23})}{2[2E_{11} + (8\nu_{12}^2 + 12\nu_{12} + 7)K_{23} + 2(\mu_{12} + \mu_{23})]^2}. \quad (6.15)
\]

### 6.2. 2D case

A system with two-dimensional (planar) random orientation of fibers in an isotropic matrix phase, appropriate to thin sections in a state of plane stress.

\[
\begin{align*}
\sigma_{11} &= \left(\frac{C_{11} - C_{12}^2}{C_{22}}\right)_{Q_{11}} \epsilon_{11} + \left(\frac{C_{12} - C_{12}C_{23}}{C_{22}}\right)_{Q_{12}} \epsilon_{22}, \\
\sigma_{22} &= \left(\frac{C_{12} - C_{12}C_{23}}{C_{22}}\right)_{Q_{12}} \epsilon_{11} + \left(\frac{C_{22} - C_{22}^2}{C_{22}}\right)_{Q_{22}} \epsilon_{22}, \\
\sigma_{12} &= 2\frac{C_{66}}{Q_{66}} \epsilon_{12}.
\end{align*}
\]

Similar to previous subsection

\[
\frac{\sigma'_{ij}}{\epsilon_{11}} \bigg|_{\text{random}} = \frac{1}{\pi} \int_0^{\pi} \frac{\sigma'_{ij}}{\epsilon_{11}} \, d\theta \quad (6.17)
\]

resulting in

\[
\begin{align*}
\frac{\sigma'_{11}}{\epsilon_{11}} \bigg|_{\text{random}} &= \frac{1}{8}(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}), \quad (6.18) \\
\frac{\sigma'_{22}}{\epsilon_{11}} \bigg|_{\text{random}} &= \frac{1}{8}(Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}). \quad (6.19)
\end{align*}
\]
Here \( \{ x_1', x_2' \} \) is a fixed coordinate system, \( x_1' \) - direction of the applied loading, while \( \{ x_1, x_2 \} \) is a random coordinate system for a fiber orientation along \( x_1 \) axis.

Since the right-hand sides of the above equations in case of effectively isotropic medium under plane stress conditions are equal to \( \frac{E}{1-\nu^2} \) and \( \frac{\nu E}{1-\nu^2} \), respectively, the planar effective isotropic Young’s modulus and Poisson’s ratio are found as

\[
E = \frac{1}{8} \left( 3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66} \right) \left( 1 - \nu^2 \right),
\]

\[
\nu = \frac{Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}}{3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}}
\]

or, related to the engineering properties, as

\[
E = \frac{1}{u_1} \left( u_1^2 - u_2^2 \right),
\]

\[
\nu = \frac{u_2}{u_1},
\]

where

\[
u_1 = \frac{3}{8} E_{11} + \frac{\mu_{12}}{2} + \frac{(3 + 2\nu_{12} + 3\nu_{12}^2)\mu_{23}K_{23}}{2(\mu_{23} + K_{23})},
\]

\[
u_2 = \frac{1}{8} E_{11} - \frac{\mu_{12}}{2} + \frac{(1 + 6\nu_{12} + \nu_{12}^2)\mu_{23}K_{23}}{2(\mu_{23} + K_{23})}.
\]

### 6.3. Asymptotic expansions

Let’s gather the formulas from the composite cylinders model.

**Uniaxial modulus** \( E_{11} \) according to Eq. 4.183:

\[
E_{11} = cE_t + (1 - c)E_M + \frac{4c(1 - c)(\nu_t - \nu_M)^2\mu_M}{k_t + \mu_M/3} + \frac{c\mu_M}{k_{St} + \mu_M/3} + 1.
\]

**Poisson’s ratio** \( \nu_{12} \) according to Eq. 4.185:

\[
\nu_{12} = c\nu_t + (1 - c)\nu_M + \frac{c(1 - c)(\nu_t - \nu_M) \left( \frac{\mu_M}{k_{St} + \mu_M/3} - \frac{\mu_M}{k_t + \mu_M/3} \right)}{k_t - k_{St} + (\mu_t - \mu_M)/3} + \frac{(1 - c)c\mu_M}{k_{St} + \mu_M/3} + 1.
\]

**Plane strain bulk modulus** \( K_{23} \) according to Eq. 4.186:

\[
K_{23} = k_M + \frac{\mu_M}{3} + \frac{c}{k_t - k_{St} + (\mu_t - \mu_M)/3} + \frac{(1 - c)c}{k_{St} + \mu_M/3}.
\]

**Shear moduli in the fiber direction** \( \mu_{12} = \mu_{31} \) according to Eq. 4.187:

\[
\frac{\mu_{12}}{\mu_M} = \frac{\mu_t(1 + c) + \mu_M(1 - c)}{\mu_t(1 - c) + \mu_M(1 + c)}.
\]
The equation for the transverse shear modulus $\mu_{23}$ (cf. subsection 4.2.3) is difficult to utilize so we use the lower bound from formula 5.25 which can be shown to be a good approximation:

$$\frac{\mu_{23}}{\mu_M} = 1 + \frac{\nu_M}{\mu - \nu_M} + \frac{c_t}{2(k_M + \nu_M^2)},$$

(6.26)

We have to substitute the above five formulas in the equations for the effective properties (either 3D or 2D case) to obtain the response of the homogenized media.

Before considering asymptotic expansions, examine first the case when both fiber and matrix are incompressible. Then

$$\mu_{3D}|_{\nu = \nu_M = 1/2} = \frac{1}{15} [E_{11} + 6(\mu_{12} + \mu_{23})],$$

(6.27)

or

$$\mu_{3D}|_{\nu = \nu_M = 1/2} = \frac{c}{5} \mu_t + \frac{1}{5} \frac{(5 + 2c + c^2)\mu + (5 + c)(1 - c)\mu_M}{(1 - c)\mu + (1 + c)\mu_M} \mu_M.$$  

(6.28)

From $E_{3D} = 3\mu_{3D}$ (true when $\nu = 1/2$, cf. subsection 4.1) follows that

$$E_{3D}|_{\nu = \nu_M = 1/2} = \frac{c}{5} E_t + \frac{1}{5} \frac{2(5 + 2c + c^2)E_t + (5 + c)(1 - c)E_M}{(1 - c)E_t + (1 + c)E_M} E_M.$$  

(6.29)

Asymptotic expansion for 3D case

First rewrite 6.14 as

$$E_{3D} = \frac{c E_t + [2 \hat{E}_{11} + (5 + 4\nu_{12} + 8\nu_{12}^2)K_{23} + 6(\mu_{12} + \mu_{23}) + O(1/cE_t)]}{6(1 + [2 \hat{E}_{11} + (7 + 12\nu_{12} + 8\nu_{12}^2)K_{23} + 2(\mu_{12} + \mu_{23})]/cE_t)},$$

(6.30)

where

$$\hat{E}_{11} = E_{11} - cE_t.$$  

(6.31)

Expanding Eq. 6.30 in series for $\frac{E_{3D}}{E_M}$ brings

$$\frac{E_{3D}}{E_M} = \frac{c E_t}{6 E_M} + \frac{\hat{E}_{11}}{6 E_M} + \frac{(3 - 4\nu_{12} + 8\nu_{12}^2)K_{23}}{12E_M} + \frac{5 \mu_{12} + \mu_{23}}{6} \frac{E_M}{E_t} + O\left(\frac{E_M}{cE_t}\right).$$

(6.32)

The above five formulas for the effective transversely isotropic media can be simplified supposing that $cE_t \gg E_M$ and $cE_t \gg k_M$ to

$$\hat{E}_{11} = (1 - c)E_M + 4c(1 - c)\mu_M \frac{(\nu_t - \nu_M)^2}{k_M + \mu_M/3} + 1,$$

$$\nu_{12} = c\nu_t + (1 - c)\nu_M + \frac{c(1 - c)(\nu_t - \nu_M)^2}{1 + \frac{c\mu_M}{k_M + \mu_M/3}},$$

$$K_{23} = \frac{k_M}{1 - c} + \frac{1 + 3c}{1 - c}\frac{\mu_M}{3},$$

$$\mu_{12} = \frac{1}{1 - c}\frac{\mu_M}{3},$$

$$\mu_{23} = \mu_M + \frac{2c(k_M + \frac{2}{3}\mu_M)}{(1 - c)(k_M + \frac{2}{3}\mu_M)} \mu_M.$$  

(6.33)
It should be noticed that expressions 6.33 are not valid for \( c = 1 \). Substituting them in Eq. 6.32 gives us the final formula. The results are rather cumbersome. Let us suppose that \( \nu_f = \nu_M = 1/4 \) in the Eqs. 6.33. Then the above equations are further simplified to

\[
\begin{align*}
\hat{E}_{11} &= (1-c)E_M, \\
\nu_{12} &= 1/4, \\
K_{23} &= \frac{2}{5} \frac{2+c}{1-c} E_M, \\
\mu_{12} &= \frac{2}{5} \frac{1+c}{1-c} E_M, \\
\mu_{23} &= \frac{1}{5} \frac{2+c}{1-c} E_M,
\end{align*}
\]

and, finally, Eq. 6.32 simplifies to the form

\[
\begin{align*}
\left. \frac{E_{2D}}{E_M} \right|_{\nu_f = \nu_M = 1/4} &= \frac{1}{3} \frac{c E_I}{E_M} + \frac{1 - c}{3} + \frac{19}{27} \frac{E_I (1+c) + E_M (1-c)}{E_I (1-c) + E_M (1+c)} + O \left( \frac{E_M}{c E_I} \right). \\
\left. \frac{E_{2D}}{E_M} \right|_{\nu_f = \nu_M = 1/2} &= \frac{1}{3} \frac{c E_I}{E_M} + \frac{1 - c}{3} + \frac{19}{27} \frac{E_I (1+c) + E_M (1-c)}{E_I (1-c) + E_M (1+c)} + O \left( \frac{E_M}{c E_I} \right).
\end{align*}
\]
7. 2D and 3D systems of platelets

2D system of platelets

For two dimensional system of platelets, the effective shear modulus can be given by

$$\mu = \frac{1}{h}(h_1\mu_1 + h_2\mu_2).$$

where $h_1$ and $h_2$ are the thickness of the platelets of different elastic constants in the matrix and $h$ is the summation of thickness of different platelets.

![Figure 7.1.: Illustration of 2D system of platelets.](image)

Under normal loading the stress fields in the composite are

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}),$$

$$\sigma_{yy} = \frac{E}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx}).$$

where $E$ is the effective Young’s modulus of the composite.

The effective Young’s modulus can be written as

$$\sigma_{xx} = \frac{1}{h}\left(h_1\frac{E_1}{1-\nu_1^2} + h_2\frac{E_2}{1-\nu_2^2}\right)(\epsilon_{xx} + \nu\epsilon_{yy}).$$

3D system of platelets

For the 3-D system of platelets, the effective Young’s modulus and effective Poisson’s ratio are

$$E = (c_1E_1 + c_2E_2) + \frac{c_1c_2E_1E_2(\nu_1^2 - \nu_2^2)}{c_1E_1(1-\nu_1^2) + c_2E_2(1-\nu_2^2)},$$

$$\nu = \frac{c_1\nu_1E_1(1-\nu_2^2) + c_2\nu_2E_2(1-\nu_1^2)}{c_1E_1(1-\nu_2^2) + c_2E_2(1-\nu_1^2)}.$$
Figure 7.2.: Illustration of 3D system of platelets.
8. Periodic Homogenization

In periodic homogenization, periodic arrangement of Representative Volume element is taken which is sufficient enough to get the response of micro-structure. If the periodically homogenized body is subjected to loading, the displacement in the composite will be equal to

\[ u_i = \langle \epsilon_{ij} \rangle X_j + \bar{u}_i + \pi_i. \]

Where \( \langle \epsilon_{ij} \rangle \) is the periodic strain or average strain. The displacement field should be \( \bar{u}_i \) periodic. The stress field can be given as

\[ \sigma_{ij} = C_{ijkl} \epsilon_{kl} = C_{ijkl} \epsilon_{kl} + C_{ijkl} \frac{\partial \bar{u}_i}{\partial X_i}. \]

It is assumed that fields vary on multiple spatial scales due to the existence of a microstructure and it is also assumed that the micro-structure is spatially periodic. The relevant field variables can be approximated by an asymptotic expansion.

\[ u_i^n(X) = u_i(X) + \eta u_i^1(X, X) + \eta^2 u_i^2(X, X) + \ldots, \]

Figure 8.1.: Illustration of periodic homogenization.

\[ \bar{u}_i = N_{i}^{kl} \tau_{kl}, \]

\[ \langle C_{ijkl} \rangle = \frac{1}{V} \int_S C_{ijmn} A_{mnkl} dV; \]

\[ A_{ijkl} = I_{ijkl} + \frac{1}{2} \left( \frac{\partial N_{ij}^{kl}}{\partial X_k} + \frac{\partial N_{ij}^{kl}}{\partial X_j} \right); \]

\[ \frac{\partial}{\partial X_j} \left( C_{ijkl} + C_{ijmn} \frac{\partial N_{kl}^{ij}}{\partial X_n} \right) = 0 \quad \text{in} \ S, \]

where \( N_{i}^{kl} \) is periodic.
Figure 8.2.: Periodic homogenization: displacement field decomposition.

References


Appendices
### A. Relations among elastic constants

<table>
<thead>
<tr>
<th></th>
<th>$E$</th>
<th>$\nu$</th>
<th>$k$</th>
<th>$\mu$</th>
<th>$\lambda$</th>
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<td>$E$</td>
<td>$\nu$</td>
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<td>$\frac{E}{2(1+\nu)}$</td>
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<td>$\mu$</td>
<td>$\frac{\mu(E-2\mu)}{3\mu - E}$</td>
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<td>$\frac{E+3\lambda+R}{6}$</td>
<td>$\frac{E-3\lambda+R}{4}$</td>
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<td>$\nu$</td>
<td>$k$</td>
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<td>$\frac{2\mu\nu}{1-2\nu}$</td>
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<td>$\frac{\lambda(1+\nu)}{3\nu}$</td>
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<td>$\frac{3k-2\mu}{6k+2\mu}$</td>
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<td>$\mu$</td>
<td>$k - \frac{2}{3}\mu$</td>
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<tr>
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<td>$\frac{\lambda}{3k-\lambda}$</td>
<td>$k$</td>
<td>$\frac{3}{2}(k - \lambda)$</td>
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<td>$\frac{3\lambda+2\mu}{3}$</td>
<td>$\mu$</td>
<td>$\lambda$</td>
</tr>
</tbody>
</table>

$R = \sqrt{E^2 + 9\lambda^2 + 2E\lambda}$
B. Symmetry properties of the stiffness (compliance) tensor

It is convenient to express the anisotropic form of Hooke’s law in matrix notation, also called Voigt notation. Voigt notation is the

\[
\begin{array}{cccccc}
ij &=& 11 & 22 & 33 & 23,32 & 13,31 & 12,21 \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\alpha &=& 1 & 2 & 3 & 4 & 5 & 6
\end{array}
\]

mapping for tensor indices for the symmetric stress and strain tensors expressed as six-dimensional vectors in an orthonormal coordinate system \((e_1, e_2, e_3)\):

\[
[\sigma] = \begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12}
\end{bmatrix}
\equiv
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{23} \\
2\varepsilon_{13} \\
2\varepsilon_{12}
\end{bmatrix}
\equiv
[\varepsilon],
\]

In case of six-dimensional stress/strain vectors presentation, the stiffness matrix is expressed as a tensor of second order

\[
c_{ijkl} =
\begin{bmatrix}
c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1131} & c_{1112} \\
c_{2211} & c_{2222} & c_{2233} & c_{2223} & c_{2231} & c_{2212} \\
c_{3311} & c_{3322} & c_{3333} & c_{3323} & c_{3331} & c_{3312} \\
c_{2311} & c_{2322} & c_{2333} & c_{2323} & c_{2331} & c_{2312} \\
c_{3111} & c_{3122} & c_{3133} & c_{3123} & c_{3131} & c_{3112} \\
c_{1211} & c_{1222} & c_{1233} & c_{1223} & c_{1231} & c_{1212}
\end{bmatrix}
\equiv
C_{ij} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}
\]

and Hooke’s law is written as

\[
[\sigma] = [C][\varepsilon] \quad \text{or} \quad \sigma_i = C_{ij}\varepsilon_j.
\]

Similarly the compliance tensor can be written as

\[
\begin{bmatrix}
s_{1111} & s_{1122} & s_{1133} & 2s_{1123} & 2s_{1131} & 2s_{1112} \\
s_{2211} & s_{2222} & s_{2233} & 2s_{2223} & 2s_{2231} & 2s_{2212} \\
s_{3311} & s_{3322} & s_{3333} & 2s_{3323} & 2s_{3331} & 2s_{3312} \\
2s_{2311} & 2s_{2322} & 2s_{2333} & 4s_{2323} & 4s_{2331} & 4s_{2312} \\
2s_{3111} & 2s_{3122} & 2s_{3133} & 4s_{3123} & 4s_{3131} & 4s_{3112} \\
2s_{1211} & 2s_{1222} & 2s_{1233} & 4s_{1223} & 4s_{1231} & 4s_{1212}
\end{bmatrix}
\equiv
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}
\end{bmatrix}
\]

As a result of the existence of a strain energy density function which satisfies

\[
\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}},
\]

the matrix \(C_{\alpha\beta}\) is symmetric. Hence, there are at most 21 different elements of \(C_{\alpha\beta}\).
Orthotropic materials have three orthogonal planes of symmetry. E.g. when the basis vectors \( e_1, e_2, e_3 \) are normals to the planes of symmetry) then

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix}
\]

A transversely isotropic material is symmetric with respect to a rotation about an axis of symmetry. For such a material, if \( \varepsilon_3 \) is the axis of symmetry, Hooke’s law can be expressed as

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1 + \nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1 + \nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + \nu) \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix}
\]

Stiffness/compliance matrix for a transversely isotropic material has 5 independent components.

Isotropic materials have 2 independent components in the stiffness/compliance matrix:

\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{13} \\
\varepsilon_{12} \\
\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
-\nu & 1 & -\nu & 0 & 0 & 0 \\
-\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2(1 + \nu) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(1 + \nu) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(1 + \nu) \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12} \\
\end{bmatrix}
\]

where \( \gamma_{ij} = 2\varepsilon_{ij} \) is the “engineering shear strain”.

The inverse relation is written as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12} \\
\end{bmatrix}
= \frac{E}{(1 + \nu)(1 - 2\nu)}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0 \\
\nu & 1 - \nu & \nu & 0 & 0 & 0 \\
\nu & -\nu & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1 - 2\nu}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 - 2\nu & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1 - 2\nu}{2} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{13} \\
\varepsilon_{12} \\
\end{bmatrix}
\]

or, using the Lamé constants, as

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{23} \\
\sigma_{13} \\
\sigma_{12} \\
\end{bmatrix}
= \begin{bmatrix}
2(\mu + \lambda) & \lambda & \lambda & 0 & 0 & 0 \\
\lambda & 2(\mu + \lambda) & \lambda & 0 & 0 & 0 \\
\lambda & \lambda & 2(\mu + \lambda) & 0 & 0 & 0 \\
0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & \mu & 0 \\
0 & 0 & 0 & 0 & 0 & \mu \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{23} \\
\varepsilon_{13} \\
\varepsilon_{12} \\
\end{bmatrix}
\]

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C. Kinematic and equilibrium equations in cylindrical and spherical coordinates

Strain-displacement relations

Cartesian coordinate
\[ e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad e_z = \frac{\partial w}{\partial z}, \]
\[ e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \quad e_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right), \quad e_{zx} = \frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right). \] (C.1)

Cylindrical coordinate
\[ \epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}, \]
\[ 2\epsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad 2\epsilon_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi \cot \theta}{r}. \] (C.2)

Spherical coordinate
\[ \epsilon_{rr} = \frac{\partial u_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \quad \epsilon_{\phi\phi} = \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} \frac{\partial u_\phi}{\partial \phi} \frac{\partial \phi}{\partial r}, \]
\[ 2\epsilon_{r\theta} = \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \quad 2\epsilon_{r\phi} = \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}. \] (C.3)

Equilibrium equations

Cartesian coordinate
\[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0, \]
\[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0, \]
\[ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial z} + F_z = 0. \] (C.4)

Cylindrical coordinate
\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{r z}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} + \rho \beta_r = 0, \]
\[ \frac{\partial \sigma_{r \theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{\partial \sigma_{r z}}{\partial z} + \frac{2\sigma_{r \theta}}{r} + \rho \beta_\theta = 0, \]
\[ \frac{\partial \sigma_{r z}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{z z}}{\partial z} + \frac{\sigma_{r z}}{r} + \rho \beta_z = 0. \] (C.5)

Spherical coordinate
\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{1}{r} \left( 2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta} \cot \theta \right) + \rho B_r = 0, \]

\[ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{1}{r} \left[ (\sigma_{\theta\theta} - \sigma_{\phi\phi}) \cot \theta + 3\sigma_{r\theta} \right] + \rho B_\theta = 0, \]  

\[ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r} \left( 3\sigma_{r\phi} + 2\sigma_{\theta\phi} \cot \theta \right) + \rho B_\phi = 0. \]  

(C.6)

Polar coordinate

\[ \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho B_r = 0, \]

\[ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} + \rho B_\theta = 0. \]  

(C.7)
D. Operators in cylindrical and spherical coordinates

Laplacian
Spherical coordinate

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}. \]  
(D.1)